

## **IV. Special Linear Programming Models**

**Some types of LP problems have a special structure and occur so frequently that we consider them separately.**

### **A. The Transportation Problem**

- **Transportation Model - a special LP model used to determine the optimal combination of paths to move a prespecified number of units from a set of sources to a set of destinations. In these problems we generally wish to minimize shipping costs or distance shipped.**

#### **Special Structure of Transportation Problems**

- **generally deal with moving units from a limited number of sources across routes to a limited number of destinations.**
- **sources have an integer-valued capacity/availability that cannot be exceeded**
- **destinations have an integer-valued capacity/demand that must be met**
- **total source supply must equal total destination demand**
- **all constraints are equalities**

**These characteristics combined to guarantee that the Transportation Problem is**

- **easy to solve quickly**
- **unimodular (which means the optimal values of the decision variables must be integer)**

### A Simple Transportation Problem Example

Suppose we need to plan daily shipments of television sets from two production facilities to three retail outlets. Note that, because they are the same size (and so cost the same amount to ship), we will not differentiate between black & white & color sets for this problem. The Tacoma production facility manufactures ten sets per day, while the Eureka production facility manufactures fifteen sets per day. We have retail outlets in San Francisco, Portland, and Seattle. Their daily demands are 5, 10, and 10, respectively. Finally, the per-unit shipment costs for each combination of production facility and retail outlet are provided in the following table:

Sources	Destinations		
	San Francisco	Portland	Seattle
Tacoma	2	4	6
Eureka	3	6	9

The appropriate formulation would be

$$\text{minimize } C = 2x_{11} + 4x_{12} + 6x_{13} + 3x_{21} + 6x_{22} + 9x_{23}$$

$$\text{subject to: } x_{11} + x_{12} + x_{13} = 10 \text{ (units available from Tacoma)}$$

$$x_{21} + x_{22} + x_{23} = 15 \text{ (units available from Eureka)}$$

$$x_{11} + x_{21} = 5 \text{ (units demanded by San Francisco)}$$

$$x_{12} + x_{22} = 10 \text{ (units demanded by Portland)}$$

$$x_{13} + x_{23} = 10 \text{ (units demanded by Seattle)}$$

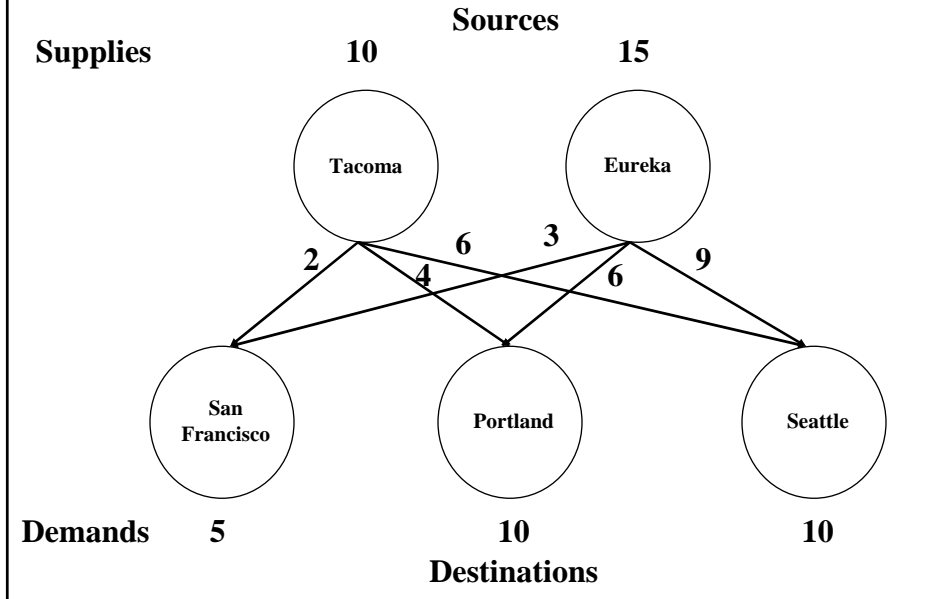
$$\text{All } x_{ij} \geq 0 \text{ (nonnegativity)}$$

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$

and  $i = 1$  (Tacoma),  $2$  (Eureka)

$j = 1$  ( San Francisco),  $2$  (Portland),  $3$  (Seattle)

A network diagram representation of the formulation would be:



A specialized Transportation Tableau representation of the formulation would be:

to from	San Francisco	Portland	Seattle	Supply
Tacoma	2	4	6	10
Eureka	3	6	9	15
Demand	5	10	10	25

One obvious feasible solution is to ship 5 units from Tacoma to San Francisco, 5 units from Tacoma to Portland, 5 units from Eureka to Portland, and 10 units from Eureka to Seattle at a total cost of  $\$2(5) + \$4(5) + \$6(5) + \$9(10) = \$150.00$ .

As with other LP algorithms, we need a feasible solution to act as a starting point. Unfortunately, it is very difficult to find an initial feasible solution to larger problems by inspection.

There are simple algorithms we can use to find a feasible solution quickly:

- Northwest Corner Method

Allocate the maximum allowable amount to the cell in the upper left-hand corner. This will deplete the supply of the first source (and leave unmet demand at the first destination) and/or satisfy the demand of the first destination (and leave unused supply at the first source).

Continue to move right and down through the tableau, making the maximum assignment at each juncture.

Northwest Corner Method - Example

to from	San Francisco	Portland	Seattle	Supply
Tacoma	<u>5</u> 2	<u>5</u> 4	6	<del>10</del>
Eureka	3	<u>5</u> 6	<u>10</u> 9	<del>15</del>
Demand	<del>5</del>	<del>10</del>	<del>10</del>	25

The feasible solution identified by the Northwest Corner Method is to ship 5 units from Tacoma to San Francisco, 5 units from Tacoma to Portland, 5 units from Eureka to Portland, and 10 units from Eureka to Seattle at a total cost of  $\$2(5) + \$4(5) + \$6(5) + \$9(10) = \$150.00$ .

**Difficulties with the Northwest Corner Method - occasionally we exhaust supply and satisfy demand simultaneously at an intermediate step:**

to from	San Francisco	Portland	Seattle	Supply
Tacoma	<b>10</b> 2	4	6	<del>10</del>
Eureka		3 6	9	15
Demand	<del>10</del>	5	10	25

**For this slightly modified version of the original transportation problem, we do exhaust supply and satisfy demand simultaneously at the first step - how do we proceed?**

**In this situation, simply move either to the cell directly to the right or below, enter a dummy shipment of zero, and proceed.**

to from	San Francisco	Portland	Seattle	Supply
Tacoma	<b>10</b> 2	<b>0</b> 4	6	<del>10</del>
Eureka		3 6	<b>10</b> 9	<del>15</del>
Demand	<del>10</del>	<del>5</del>	<del>10</del>	25

**This approach will eventually push the problem forward so that we can find a feasible solution.**

**- Cheapest Cell Starting Method**

Allocate the maximum allowable amount to the cell with the lowest cost.

Eliminate the source and/or destination that is exhausted or satisfied by this entry.

Allocate the maximum allowable amount to the remaining cell with the lowest cost.

Eliminate the source(s) and/or destination(s) that is(are) exhausted or satisfied by this entry.

Continue this procedure until all sources and destinations are satisfied.

Because objective function coefficients are considered explicitly, This method *often* provides an initial solution that is superior to that provided by the Northwest Corner Method.

**Cheapest Cell Starting Method - Example**

to from	San Francisco	Portland	Seattle	Supply
Tacoma	<span style="border: 1px solid black; border-radius: 50%; padding: 2px;">5</span> <span style="border: 1px solid black; padding: 2px;">2</span>	<span style="border: 1px solid black; border-radius: 50%; padding: 2px;">5</span> <span style="border: 1px solid black; padding: 2px;">4</span>	<span style="border: 1px solid black; padding: 2px;">6</span>	<del>10</del>
Eureka	<span style="border: 1px solid black; padding: 2px;">3</span>	<span style="border: 1px solid black; border-radius: 50%; padding: 2px;">5</span> <span style="border: 1px solid black; padding: 2px;">6</span>	<span style="border: 1px solid black; border-radius: 50%; padding: 2px;">10</span> <span style="border: 1px solid black; padding: 2px;">9</span>	<del>15</del>
Demand	<del>5</del>	<del>10</del>	<del>10</del>	25

In this case the Cheapest Cell Starting Method yields the same feasible solution as the identified by the Northwest Corner Method.

- Vogel's Approximation Method (VAM)

For each unassigned row and column (corresponding to unexhausted supply or unmet demand) calculate the difference between the cheapest and second cheapest cells.

Identify the row or column with the greatest difference and allocate the maximum allowable amount to the cheapest cell in that row/column.

Eliminate the source(s) and/or destination(s) that is(are) exhausted or satisfied by this entry.

Continue this procedure until all sources and destinations are satisfied.

Because it takes marginal differences in objective function coefficients into account explicitly, this method *often* provides an initial solution that is superior to that provided by the Cheapest Cell Starting Method.

Vogel's Approximation Method - Example

to from	San Francisco	Portland	Seattle	Supply	
Tacoma	2	4	10	6	10
Eureka	3	6		9	15
Demand	5	10	10		25

$3-2=1$                        $6-4=2$                        $9-6=3$

$4-2=2$   
 $6-3=3$

In this case the we could start with the Eureka row or the Seattle column - we arbitrarily choose to start with the Seattle Column and allocate 10 units to the cheapest Seattle route (Tacoma to Seattle).

At this point Tacoma is exhausted and Seattle is satisfied and so taken out of consideration:

to from	San Francisco	Portland	Seattle	Supply
Tacoma	2	4	10	16
Eureka	5	3	10	15
Demand	5	10	10	25

We are only left with two cells (both in the same row) - we can make these assignments (5 units from Eureka to San Francisco and 10 units from Eureka to Portland) without any further analytic assistance. The total cost of this feasible solution is  $\$6(10) + \$3(5) + \$6(10) = \$135.00$ .

Note that:

- The number of unassigned cells in any feasible solution is equal to  

$$\text{number of sources} + \text{number of destinations} - 1$$
- Each of these methods is a heuristic and so does not guarantee the optimal allocation - just feasibility. As with other LP algorithms, we need a feasible solution to begin with. Unfortunately, it is very difficult to find an initial feasible solution to larger problems by inspection.

Once we have a feasible solution we can use the *Simplex Transportation Algorithm* to find the optimal allocation.

**- Simplex Transportation (Modified Distribution or MODI) Algorithm (for a *minimization problem*)**

**Identify a starting point (feasible solution)**

**Identify the entering cell**

denote the unit cost for the cell in the  $i^{\text{th}}$  row (source) and  $j^{\text{th}}$  column (destination) as  $c_{ij}$

calculate row number ( $r_i$ ) and column numbers ( $k_j$ ) that satisfy  $c_{ij} = r_i + k_j$  for each basic (nonempty) cell (start by arbitrarily assigning  $r_i = 0$  and working forward)

calculate the Improvement Difference ( $c_{ij} - r_i - k_j$ ) for each unassigned cell

the unassigned cell with the greatest absolute negative Improvement Difference is the entering cell in the next iteration

**Identify the exiting cell**

form the unique closed-loop path of horizontal and vertical directions with the entering cell at one corner and basic cells at all other corners

assign a positive sign (+) to the corner of the closed-loop path to the entering cell then alternate assigning negative (-) and positive (+) signs at adjacent corners of the closed-loop path (note that shifting one unit into each cell with a + and one unit out of each cell with a - will preserve feasibility and improve the objective function)

**Shift units to find a new feasible solution**

shift the maximum number of units feasible into each cell with a + and out of each cell with a - and recalculate the value of the objective function.

Repeat until no negative Improvement Differences ( $c_{ij} - r_i - k_j$ ) exist at any unassigned cells

**Example - Use the results of the NW Corner Method for the initial solution (with an objective function value of at a total cost of  $\$2(5) + \$4(5) + \$6(5) + \$9(10) = \$150.00$ ).**

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	5 2	5 4	6	10	0
Eureka	3	5 6	10 9	15	$6=r_2+4 \rightarrow$ $r_2=2$
Demand	5	10	10	25	
$k_j$	$2=0+k_1 \rightarrow$ $k_1=2$	$4=0+k_2$ $\rightarrow k_2=4$	$9=2+k_3$ $\rightarrow k_3=7$		

Here we have calculated the row numbers ( $r_i$ ) and column numbers ( $k_j$ ) that satisfy  $c_{ij} = r_i + k_j$  for each basic cell (starting by arbitrarily assigning  $r_1 = 0$ ).

Now we calculate the Improvement Difference ( $c_{ij} - r_i - k_j$ ) for each unassigned cell.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	$\textcircled{5}$ 2	$\textcircled{5}$ 4	6	10	$r_1=0$
Eureka	3	$\textcircled{5}$ 6	$\textcircled{10}$ 9	15	$r_2=2$
Demand	5	10	10	25	

$k_j$                        $k_1=2$                        $k_2=4$                        $k_3=7$

We have a tie for the unassigned cell with the greatest absolute negative Improvement Difference (cells 1,3 and 2,1) - we arbitrarily choose to enter cell 1,3 in the next iteration.

Form the unique closed-loop path of horizontal and vertical directions with the entering cell at one corner and basic cells at all other corners.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	$\textcircled{5}$ 2	$\textcircled{5}$ 4	6	10	
Eureka	3	$\textcircled{5}$ 6	$\textcircled{10}$ 9	15	
Demand	5	10	10	25	

$k_j$

shift the maximum number of units feasible into each cell with a + and out of each cell with a - and recalculate the value of the objective function.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	5	5	6	10	
Eureka		5	10	15	
Demand	5	10	10	25	

$k_j$

We can only shift 5 on our closed-loop path (why?)

Here are the final result of our first iteration (with an objective function value of at a total cost of  $\$2(5) + \$6(10) + \$6(5) + \$9(5) = \$145.00$ ):

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	5		5	10	
Eureka		10	5	15	
Demand	5	10	10	25	

$k_j$

Now we need to recalculate the Improvement Differences ( $c_{ij} - r_i - k_j$ ) and check to see if any negative values exist at any unassigned cells

Here we have calculated the row numbers ( $r_i$ ) and column numbers ( $k_j$ ) that satisfy  $c_{ij} = r_i + k_j$  for each basic cell (starting by arbitrarily assigning  $r_1 = 0$ ).

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	5   2	4   5   6	10	0	
Eureka	3	10   6   5   9	15	$9=r_2+6 \rightarrow r_2=3$	
Demand	5	10	10	25	
$k_j$	$2=0+k_1 \rightarrow k_1=2$	$6=3+k_2 \rightarrow k_2=3$	$6=0+k_3 \rightarrow k_3=6$		

Now we use these values to recalculate the Improvement Differences ( $c_{ij} - r_i - k_j$ ):

Here are the recalculated Improvement Differences ( $c_{ij} - r_i - k_j$ ) for the unassigned cells.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	5   2	4   5   6	10	$r_1=0$	
Eureka	3	10   6   5   9	15	$r_2=3$	
Demand	5	10	10	25	
$k_j$	$k_1=2$	$k_2=3$	$k_3=6$		

$c_{12}-r_1-k_2=4-0-3=1$   
 $c_{21}-r_2-k_1=3-3-2=-2$

the unassigned cell with the greatest absolute negative Improvement Difference (cell 2,1) is the entering cell in the next iteration

Form the unique closed-loop path of horizontal and vertical directions with the entering cell at one corner and basic cells at all other corners.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	- 5 2	4	+ 5 6	10	
Eureka	+ 3	10 6	- 5 9	15	
Demand	5	10	10	25	

$k_j$

shift the maximum number of units feasible into each cell with a + and out of each cell with a - and recalculate the value of the objective function.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	- 5 2	4	+ 5 6	10	
Eureka	+ 3	10 6	- 5 9	15	
Demand	5	10	10	25	

$k_j$

Again we can only shift 5 on our closed-loop path.

Here are the final result of our second iteration (with an objective function value of at a total cost of  $\$3(5) + \$6(10) + \$6(10) = \$135.00$ ):

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	$\textcircled{0}$   2	4	$\textcircled{10}$   6	10	
Eureka	$\textcircled{5}$   3	$\textcircled{10}$   6	9	15	
Demand	5	10	10	25	

$k_j$

Note that two cells exited on this iteration - to continue, make either one a basic variable with an assigned value of zero.

Here we have calculated the row numbers ( $r_i$ ) and column numbers ( $k_j$ ) that satisfy  $c_{ij} = r_i + k_j$  for each basic cell (starting by arbitrarily assigning  $r_1 = 0$ ).

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	$\textcircled{0}$   2	4	$\textcircled{10}$   6	10	0
Eureka	$\textcircled{5}$   3	$\textcircled{10}$   6	9	15	$3=r_2+2 \rightarrow$ $r_2=1$
Demand	5	10	10	25	

$k_j$        $2=0+k_1 \rightarrow$   
 $k_1=2$        $6=1+k_2$   
 $\rightarrow k_2=5$        $6=0+k_3$   
 $\rightarrow k_3=6$

Now we use these values to recalculate the Improvement Differences ( $c_{ij} - r_i - k_j$ ):

Here are the recalculated Improvement Differences ( $c_{ij} - r_i - k_j$ ) for the unassigned cells.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	<b>0</b> 2	4 <b>10</b> 6	10	$r_1=0$	
Eureka	<b>5</b> 3	<b>10</b> 6	9 15	$r_2=1$	
Demand	5	10	10	25	
	$k_j$	$k_1=2$	$k_2=5$	$k_3=6$	

$c_{12} - r_1 - k_2 = 4 - 0 - 5 = -1$   
 $c_{23} - r_2 - k_3 = 9 - 1 - 6 = 2$

the unassigned cell with the greatest absolute negative Improvement Difference (cell 1,2) is the entering cell in the next iteration

Form the unique closed-loop path of horizontal and vertical directions with the entering cell at one corner and basic cells at all other corners.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	- <b>0</b> 2	+ 4 <b>10</b> 6	10		
Eureka	+ <b>5</b> 3	- <b>10</b> 6	9 15		
Demand	5	10	10	25	
	$k_j$				

Now shift the maximum number of units feasible into each cell with a + and out of each cell with a - and recalculate the value of the objective function.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	- 0 2	+ 4 10	6	10	
Eureka	+ 5 3	- 10 6	9	15	
Demand	5	10	10	25	

$k_j$

Now we can only shift 0 on our closed-loop path.

Here are the final result of our third iteration (with an objective function value of at a total cost of  $\$3(5) + \$6(10) + \$6(10) = \$135.00$ ):

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	2	0 4	10 6	10	
Eureka	5 3	10 6	9	15	
Demand	5	10	10	25	

$k_j$

Now we need to recalculate the Improvement Differences ( $c_{ij} - r_i - k_j$ ) and check to see if any negative values exist at any unassigned cells (Note that two cells exited on this iteration - to continue, make either one a basic variable with an assigned value of zero).

Here we have calculated the row numbers ( $r_i$ ) and column numbers ( $k_j$ ) that satisfy  $c_{ij} = r_i + k_j$  for each basic cell (starting by arbitrarily assigning  $r_1 = 0$ ).

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	2	0	4	10	0
Eureka	5	10	6	9	$6=r_2+4 \rightarrow r_2=2$
Demand	5	10	10	25	

$k_j$        $3=2+k_1 \rightarrow k_1=1$        $4=0+k_2 \rightarrow k_2=4$        $6=0+k_3 \rightarrow k_3=6$

Now we use these values to recalculate the Improvement Differences ( $c_{ij} - r_i - k_j$ ):

Here are the recalculated Improvement Differences ( $c_{ij} - r_i - k_j$ ) for the unassigned cells.

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	2	0	4	10	$r_1=0$
Eureka	5	10	6	9	$r_2=2$
Demand	5	10	10	25	

$k_j$        $k_1=1$        $k_2=4$        $k_3=6$

$c_{11}-r_1-k_1=2-0-1=1$        $c_{23}-r_2-k_3=9-2-6=1$

No unassigned cell has a negative Improvement Difference, so we are at optimality (ship 5 units from Eureka to San Francisco, 10 units from Tacoma to Seattle, and 10 units from Eureka to Portland at a total cost of \$135.00).

How would this algorithm differ if we were solving a maximization problem?

Simple - we would look for positive (instead of negative) Improvement Differences ( $c_{ij} - r_i - k_j$ ) in the unassigned cells.

- This is great when supply = demand - what do we do when there is a discrepancy?

Consider a slightly modified version of our example:

to from	San Francisco	Portland	Seattle	Supply
Tacoma	2	4	6	10
Eureka	3	6	9	15
Demand	5	5	10	20\25

Here supply exceeds demand by 5 units - we need to create a destination that can *absorb* the five units that are not in demand. We will add a Dummy Destination with a demand of 5 units to balance supply and demand:

Now supply = demand, and we can use the techniques we have already developed to solve this transportation problem.

to from	San Francisco	Portland	Seattle	Dummy	Supply
Tacoma	2	4	6	0	10
Eureka	3	6	9	0	15
Demand	5	5	10	5	25

What is the economic interpretation of the Dummy Destination? Why do its cells have objective function coefficients ( $c_{ij}$ 's) of zero?

How to handle supply < demand in the transportation model?

Consider a slightly modified version of our example:

to from	San Francisco	Portland	Seattle	Supply
Tacoma	2	4	6	10
Eureka	3	6	9	5
Demand	5	10	10	25\15

Here demand exceeds supply by 10 units - we need to create a source that can *absorb* the ten unavailable units. We will add a Dummy Source with a supply of 10 units to balance supply and demand:

Now supply = demand, and we can again use the techniques we have already developed to solve this transportation problem.

to from	San Francisco	Portland	Seattle	Supply
Tacoma	2	4	6	10
Eureka	3	6	9	5
Dummy	0	0	0	10
Demand	5	10	10	25

What is the economic interpretation of the Dummy Destination? Why do its cells have objective function coefficients ( $c_{ij}$ 's) of zero?

- How do we handle an unallowable route (impossible cell)?

Suppose that California will not accept shipments from Washington. How do we modify our problem?

to from	San Francisco	Portland	Seattle	Supply
Tacoma	M	4	6	10
Eureka	3	6	9	15
Demand	5	10	10	25

Because this is a minimization problem, we assign any unacceptable route (impossible cell) a large positive objective function coefficient ( $c_{ij}$ ). What would we do if this were a maximization problem?

**One appropriate formulation would be**

$$\text{minimize } C = Mx_{11} + 4x_{12} + 6x_{13} + 3x_{21} + 6x_{22} + 9x_{23}$$

$$\text{subject to: } x_{11} + x_{12} + x_{13} = 10 \text{ (units available from Tacoma)}$$

$$x_{21} + x_{22} + x_{23} = 15 \text{ (units available from Eureka)}$$

$$x_{11} + x_{21} = 5 \text{ (units demanded by San Francisco)}$$

$$x_{12} + x_{22} = 10 \text{ (units demanded by Portland)}$$

$$x_{13} + x_{23} = 10 \text{ (units demanded by Seattle)}$$

$$\text{All } x_{ij} \geq 0 \text{ (nonnegativity)}$$

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$

and  $i = 1$  (Tacoma),  $2$  (Eureka)

$j = 1$  (San Francisco),  $2$  (Portland),  $3$  (Seattle)

**Another appropriate formulation would be**

$$\text{minimize } C = 2x_{11} + 4x_{12} + 6x_{13} + 3x_{21} + 6x_{22} + 9x_{23}$$

$$\text{subject to: } x_{11} + x_{12} + x_{13} = 10 \text{ (units available from Tacoma)}$$

$$x_{21} + x_{22} + x_{23} = 15 \text{ (units available from Eureka)}$$

$$x_{11} + x_{21} = 5 \text{ (units demanded by San Francisco)}$$

$$x_{12} + x_{22} = 10 \text{ (units demanded by Portland)}$$

$$x_{13} + x_{23} = 10 \text{ (units demanded by Seattle)}$$

$$x_{11} = 0 \text{ (no allowable shipments from Tacoma to San Francisco)}$$

$$\text{All } x_{ij} \geq 0 \text{ (nonnegativity)}$$

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$

and  $i = 1$  (Tacoma),  $2$  (Eureka)

$j = 1$  (San Francisco),  $2$  (Portland),  $3$  (Seattle)

Still another appropriate formulation would be

minimize  $C=4x_{12} + 6x_{13} + 3x_{21} + 6x_{22} + 9x_{23}$

subject to:  $x_{12} + x_{13} = 10$  (units available from Tacoma)  
 $x_{21} + x_{22} + x_{23} = 15$  (units available from Eureka)  
 $x_{21} = 5$  (units demanded by San Francisco)  
 $x_{12} + x_{22} = 10$  (units demanded by Portland)  
 $x_{13} + x_{23} = 10$  (units demanded by Seattle)  
 All  $x_{ij} \geq 0$  (nonnegativity)

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$   
 and  $i = 1$  (Tacoma),  $2$  (Eureka)  
 $j = 1$  ( San Francisco),  $2$  (Portland),  $3$  (Seattle)

- How do we handle an upper limited (capacitated) cell?

Suppose that Portland will accept no more than three units from Tacoma. How do we modify our problem?

Upper cell bound of 3 units	to	San Francisco	Portland	Seattle	Supply
	from				
Tacoma		2	4	6	10
Eureka		3	6	9	15
Demand		5	10	10	25

Because this is a minimization problem, we assign any unacceptable route (impossible cell) a large positive objective function coefficient ( $c_{ij}$ ). What would we do if this were a maximization problem?

The appropriate formulation would be

minimize  $C = 2x_{11} + 4x_{12} + 6x_{13} + 3x_{21} + 6x_{22} + 9x_{23}$

subject to:  $x_{11} + x_{12} + x_{13} = 10$  (units available from Tacoma)

$x_{21} + x_{22} + x_{23} = 15$  (units available from Eureka)

$x_{11} + x_{21} = 5$  (units demanded by San Francisco)

$x_{12} + x_{22} = 10$  (units demanded by Portland)

$x_{13} + x_{23} = 10$  (units demanded by Seattle)

$x_{12} \leq 3$  (no more than three units shipped from Tacoma to Portland)

All  $x_{ij} \geq 0$  (nonnegativity)

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$

and  $i = 1$  (Tacoma),  $2$  (Eureka)

$j = 1$  (San Francisco),  $2$  (Portland),  $3$  (Seattle)

- For example, to place a three unit limit on shipments to Portland from Tacoma:

Of course, this may cause some simple problems with our heuristics - consider the NW Corner method:

to from	San Francisco	Portland	Seattle	Supply
Tacoma	5 2	M 2 6	10-3=7	
Tacoma	0 2	3 4	6	3
Eureka	3	7 6	8 9	15
Demand	5	10	10	25

We can use our usual trick to get around the lack of a valid move out of the Tacoma-SF cell, and some common sense after we reach the SW corner and fail to exhaust Eureka or satisfy Seattle.

We could alternatively place a three unit limit on shipments to Portland from Tacoma in this manner:  
Of course, this again may cause some simple problems with our heuristics - consider the NW Corner method:

to from	San Francisco	Portland	Portland	Seattle	Supply
Tacoma	5 2	M	4	5 6	10
Eureka	0 3	7 6	3 6	5 9	15
Demand	5	10-3=7	3	10	25

We can again use our usual trick to get around the lack of a valid move out of the Tacoma-SF cell, and some common sense after we reach the SW corner and fail to exhaust Eureka or satisfy Seattle.

- How do we spot Alternate Optimal Solutions?

Look for an empty cell with an Improvement Difference of zero ( $c_{ij} - r_i - k_j = 0$ )

to from	San Francisco	Portland	Seattle	Supply	$r_i$
Tacoma	2 $c_{11}-r_1-k_1=2-0-1=1$	0 4	10 6	10	$r_1=0$
Eureka	5 3	10 6	8 $c_{23}-r_1-k_3=8-2-6=0$	15	$r_2=2$
Demand	5	10	10	25	

$k_j$                        $k_1=1$                        $k_2=4$                        $k_3=6$

In this slightly modified version of our example problem, we have no negative improvement ratios - but one improvement ratio is zero (indicating its per unit impact on the objective function would be neutral).

## **B. The Transshipment Problem**

- **Transshipment Model** - a special type of transportation problem for which the network is not bipartite - used to determine the optimal combination of routes to move a prespecified number of units from a set of sources *through a set of intermediary sites* to a set of destinations. In these problems we again generally wish to minimize shipping costs or distance shipped.

### **A Simple Transshipment Problem Example**

Suppose we need to plan daily shipments of television sets from two production facilities to three retail outlets. Note that, because they are the same size (and so cost the same amount to ship), we will not differentiate between black & white & color sets for this problem. The Tacoma production facility manufactures ten sets per day, while the Eureka production facility manufactures fifteen sets per day. We have retail outlets in San Francisco, Portland, and Seattle. Their daily demands are 5, 10, and 10, respectively.

Shipments do not go directly from the two production facilities to the three retail outlets, but are stored in warehouses located in Eugene and Olympia. The Eugene warehouse has a capacity of 8 units while the Olympia warehouse has a capacity of 20 units.

Finally, the per-unit shipment costs (for each combination of warehouse and production facility or retail outlet) are provided in the following tables:

Warehouses		
Sources	Eugene	Olympia
Tacoma	4	2
Eureka	5	7

Destinations			
Warehouses	San Francisco	Portland	Seattle
Eugene	6	2	5
Olympia	9	4	3

Since this is a special type of Transportation Problem (which is a special type of LP Problem) this (or any) transshipment problem can be formulated as an LP;

$$\text{minimize } C = 4x_{11} + 2x_{12} + 5x_{21} + 7x_{22} + 0x_{31} + 6x_{33} + 2x_{34} + 5x_{35} + 0x_{42} + 9x_{43} + 4x_{44} + 3x_{45}$$

subject to:

$$x_{11} + x_{12} = 10 \text{ (units available from Tacoma)}$$

$$x_{21} + x_{22} = 15 \text{ (units available from Eureka)}$$

$$x_{11} + x_{21} \leq 8 \text{ (Eugene inflow)}$$

$$x_{12} + x_{22} \leq 20 \text{ (Olympia inflow)}$$

$$x_{31} + x_{33} + x_{34} + x_{35} \leq x_{11} + x_{21} \text{ (Eugene outflow)}$$

$$x_{42} + x_{43} + x_{44} + x_{45} \leq x_{12} + x_{22} \text{ (Olympia outflow)}$$

$$x_{33} + x_{43} = 5 \text{ (unit demand at San Francisco)}$$

$$x_{34} + x_{44} = 10 \text{ (unit demand at Portland)}$$

$$x_{35} + x_{45} = 10 \text{ (unit demand at Seattle)}$$

$$\text{All } x_{ij} \geq 0 \text{ (nonnegativity)}$$

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$   
and  $i = 1$  (Tacoma),  $2$  (Eureka),  $3$  (Eugene),  $4$  (Olympia)  
 $j = 1$  (Eugene),  $2$  (Olympia),  $3$  (San Francisco),  $4$  (Portland),  $5$  (Seattle)

Notice that the intermediate locations (warehouses in this model) function as both sources and destinations.

This cleaner looking formulation (although mathematically equivalent) formulation is given more frequently:

$$\text{minimize } C = 4x_{11} + 2x_{12} + 5x_{21} + 7x_{22} + 0x_{31} + 6x_{33} + 2x_{34} + 5x_{35} + 0x_{42} + 9x_{43} + 4x_{44} + 3x_{45}$$

$$\text{subject to: } x_{11} + x_{12} = 10 \text{ (units available from Tacoma)}$$

$$x_{21} + x_{22} = 15 \text{ (units available from Eureka)}$$

$$x_{11} + x_{21} \leq 8 \text{ (Eugene inflow)}$$

$$x_{12} + x_{22} \leq 20 \text{ (Olympia inflow)}$$

$$-x_{11} - x_{21} + x_{31} + x_{33} + x_{34} + x_{35} \leq 0 \text{ (Eugene outflow)}$$

$$-x_{12} - x_{22} + x_{42} + x_{43} + x_{44} + x_{45} \leq 0 \text{ (Olympia outflow)}$$

$$x_{33} + x_{43} = 5 \text{ (unit demand at San Francisco)}$$

$$x_{34} + x_{44} = 10 \text{ (unit demand at Portland)}$$

$$x_{35} + x_{45} = 10 \text{ (unit demand at Seattle)}$$

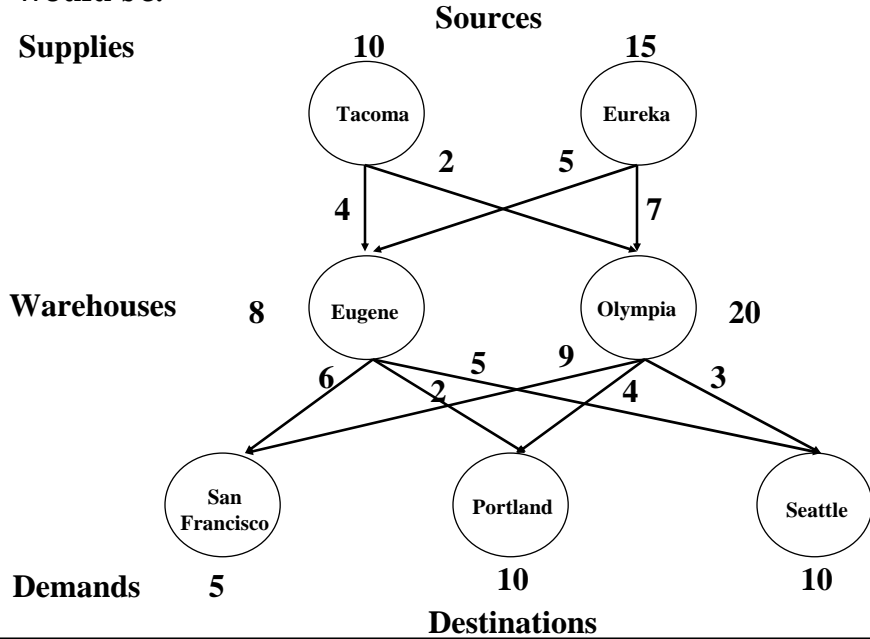
$$\text{All } x_{ij} \geq 0 \text{ (nonnegativity)}$$

Where  $x_{ij}$  is the number of units shipped from source  $i$  to destination  $j$

and  $i = 1$  (Tacoma),  $2$  (Eureka),  $3$  (Eugene),  $4$  (Olympia)

$j = 1$  (Eugene),  $2$  (Olympia),  $3$  (San Francisco),  $4$  (Portland),  $5$  (Seattle)

A network diagram representation of the formulation would be:



The specialized Transportation Tableau can be used to represent of the formulation:

to from	Eugene	Olympia	San Francisco	Portland	Seattle	Supply
Tacoma	2	4	M	M	M	10
Eureka	5	7	M	M	M	15
Eugene	0	M	6	2	5	8
Olympia	M	0	9	4	3	20
Demand	8	20	5	10	10	53

From this point we can use the methodologies we have already developed for solving transportation problems.

**Note that:**

- We have not allowed for intra-warehouse transfers (these could be viable in some problems)
- We have not allowed for direct shipping from original sources to final destinations (these could be viable in some problems)
- The two intermediary sites (warehouses) have excess capacity:
  - Will this be used in this problem?
  - What is the economic interpretation of such units?

## **C. The Assignment Problem**

1. Assignment Model - a special type of transportation problem used to determine the optimal combination of routes to move a prespecified number of units from a set of sources (each of which has *exactly one unit available*) to a set of destinations (each of which *desires exactly one unit*). In these problems we again generally wish to minimize costs.

### A Simple Assignment Problem Example

Suppose we have three Project Director (Albert, Bonnie, and Clyde) and three clients/projects (Xanadu, Inc., Yasmine Ltd., and Zephyr Co.). Each Project Director can be assigned to only one client/project. Given the estimated time (in hours) it would take each Project Director to complete each project, how would you assign the projects to the Project Directors in a manner that minimizes the total time to complete the three projects?

Project Leader	Client/Project		
	Xanadu, Inc.	Yasmine LTD.	Zephyr Co.
Albert	10	15	9
Bonnie	9	18	5
Clyde	6	14	3

The appropriate formulation would be

$$\text{minimize } C = 10x_{11} + 15x_{12} + 9x_{13} + 9x_{21} + 18x_{22} + 5x_{23} + 6x_{31} + 14x_{32} + 3x_{33}$$

$$\text{subject to: } x_{11} + x_{12} + x_{13} = 1 \text{ (assignment to Albert)}$$

$$x_{21} + x_{22} + x_{23} = 1 \text{ (assignment to Bonnie)}$$

$$x_{31} + x_{32} + x_{33} = 1 \text{ (assignment to Clyde)}$$

$$x_{11} + x_{21} + x_{31} = 1 \text{ (assignment of Xanadu)}$$

$$x_{12} + x_{22} + x_{32} = 1 \text{ (assignment of Yasmine)}$$

$$x_{13} + x_{23} + x_{33} = 1 \text{ (assignment of Zephyr)}$$

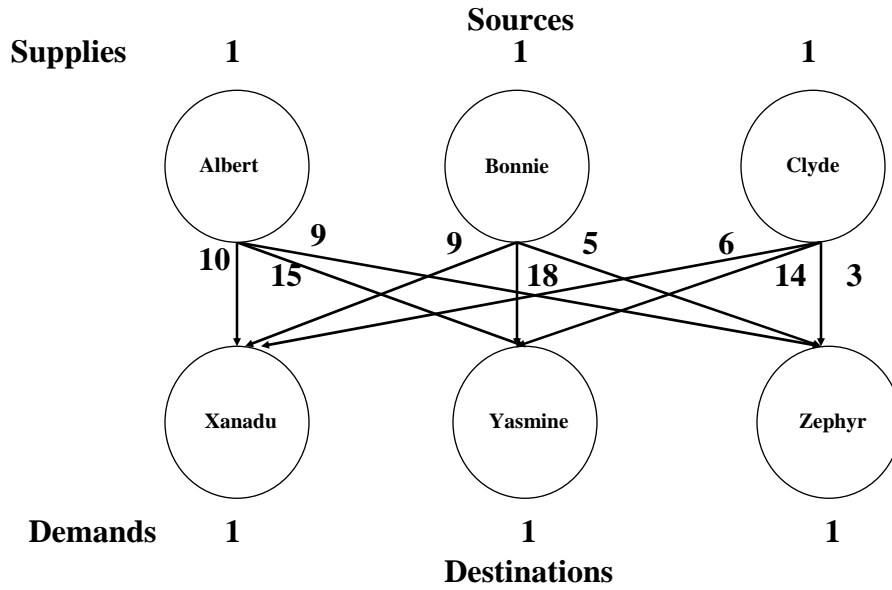
$$\text{All } x_{ij} \geq 0 \text{ (nonnegativity)}$$

Where  $x_{ij}$  indicates if Project Director  $i$  is assigned to Client/Project  $j$

and  $i = 1$  (Albert),  $2$  (Bonnie),  $3$  (Clyde)

$j = 1$  (Xanadu, Inc.),  $2$  (Yasmine Ltd.),  $3$  (Zephyr Co.)

A network diagram representation of the formulation would be:



Because the Assignment Problem is a type of Transportation Problem, we could represent this problem with a Transportation Tableau:

task \ PD	Xanadu	Yasmine	Zephyr	Supply
Albert	10	15	9	1
Bonnie	9	18	5	1
Clyde	6	14	3	1
Demand	1	1	1	3

At this point we could solve the Assignment Problem just as we would any other Transportation Problem - but there is an easier way:

- **The Hungarian Algorithm**

**The TRANSFORMATION PHASE**

- Find the lowest cost cell in a particular row and subtract that amount from every cell in the row. Repeat for each row. These are referred to as the *Reduced Costs*.
- Using the Reduced Costs calculated in the previous step, find the smallest value in a particular column and subtract that amount from every cell in the column. Repeat for each column.
- Find the column or row containing the greatest number of zeros. Draw a line through that row or column. Repeat this process until all zeros are covered.

- If the number of lines exactly equals the number of rows (or number of columns) you are at optimality. Move to the SEARCH PHASE.
- If the number of lines is less than the number of rows (or number of columns) find the minimum value uncovered by any lines and subtract that value from all uncovered cells. Return to Step #3 (covering all zeros with the minimum number of horizontal and vertical lines necessary).

**The SEARCH PHASE**

- Copy the final transformed matrix onto a fresh matrix.
- Identify a row or column with exactly one zero - circle that element and draw a line through both the row and column to which this element belongs. Repeat until every row and column has either been crossed out or has multiple uncovered zero elements.

- Circled elements represent optimal assignments. Judiciously make the other assignments to correspond to other (uncovered) zeros so that each task and each individual are each assigned in a one-to-one basis.
- Compute the total cost of the assignment.

Consider our previous example:

task \ PD	Xanadu	Yasmine	Zephyr
Albert	10	15	9
Bonnie	9	18	5
Clyde	6	14	3

Find the lowest cost cell in a particular row and subtract that amount from every cell in the row. Repeat for each row. These are referred to as the *Reduced Costs*.

These are referred to as the *Reduced Costs*.

task \ PD	Xanadu	Yasmine	Zephyr	Row Min
Albert	$10-9=1$	$15-9=6$	$9-9=0$	9
Bonnie	$9-5=4$	$18-5=13$	$5-5=0$	5
Clyde	$6-3=3$	$14-3=11$	$3-3=0$	3

Using these Reduced Costs, now find the smallest value in a particular column and subtract that amount from every cell in the column. Repeat for each column.

Here are the results:

task \ PD	Xanadu	Yasmine	Zephyr	Column Min
Albert	$1-1=0$	$6-6=0$	$0-0=0$	1
Bonnie	$4-1=3$	$13-6=7$	$0-0=0$	6
Clyde	$3-1=2$	$11-6=5$	$0-0=0$	0

Now find the column or row containing the greatest number of zeros (row one or column three in this case) and draw a line through that row or column. Repeat this process until all zeros are covered.

We can choose either row one or column three - we arbitrarily start with row 1.

task \ PD	Xanadu	Yasmine	Zephyr
Albert	0	0	0
Bonnie	3	7	0
Clyde	②	5	0

We now observe that a line drawn through column 3 would cover all zero elements. Thus the number of lines (2) is less than the number of rows or number of columns (3), so we find the minimum value uncovered by any lines and subtract that value from all uncovered cells. We then return to Step #3.

Again we can choose either row one or column three - we arbitrarily start with row 1.

task \ PD	Xanadu	Yasmine	Zephyr
Albert	0	0	0
Bonnie	1	5	0
Clyde	0	3	0

We now observe that a line drawn through column 3 would cover all zero elements. At this point we still must draw a line (through column 1 or row three) to cover all zero elements. Thus the number of lines (3) is equal to the number of rows or number of columns (3), so we are at optimality - go to the Search Phase!.

Identify a row or column with exactly one zero (either row two or column three in this case)

task PD	Xanadu	Yasmine	Zephyr
Albert	0	0	0
Bonnie	1	5	0
Clyde	0	3	0

Now circle that element and draw a line through both the row and column to which this element belongs. Repeat until every rows and columns have either been crossed out or has multiple uncovered zero elements.

Now we can make the optimal assignments:

task PD	Xanadu	Yasmine	Zephyr
Albert	0	0	0
Bonnie	1	5	0
Clyde	0	3	0

Project Director	Client/Project	Assignment Cost
Clyde	Xanadu, Inc.	6
Albert	Yasmine Ltd.	15
Bonnie	Zephyr Co.	5

For a total cost of \$26.00.

- This is great when the number of tasks and individuals are equal - what do we do when there is a discrepancy?

Create a dummy task or dummy individual (with a cost of zero for each corresponding assignment) just as we did with Transportation and Transshipment problems.

- Can we handle a problem in which one of the individuals can handle two (or more) tasks? How about one task that requires two (or more) individuals?

Absolutely - just break the individual (or task) into multiples, each of which can accept one assignment (I.e., if Bonnie could do two tasks, create a Bonnie1 and a Bonnie2)

- How do we handle an unallowable assignment?

Just as in Transportation - give the assignment a very undesirable objective function coefficient (M or -M).

- How do we handle maximization problems?

One of two ways:

Multiply all objective function coefficients by -1 and minimize

or

Before you start, find the largest element in a particular column, then subtract every element in that column from this value. Repeat for every column.

task \ PD	Xanadu	Yasmine	Zephyr
Albert	10	15	9
Bonnie	9	18	5
Clyde	6	14	3
Column Max	10	18	9

Now start the Hungarian algorithm from this point - the result will be the assignments that maximize the objective function.

task \ PD	Xanadu	Yasmine	Zephyr
Albert	$10-10=0$	$18-15=3$	$9-9=0$
Bonnie	$10-9=1$	$18-18=0$	$9-5=4$
Clyde	$10-6=4$	$18-14=4$	$9-3=6$
Column Max	10	18	9