

VI. Project Planning & Scheduling

Here we are concerned with analytic techniques to aid in the coordination of numerous separate jobs/activities that comprise a larger overall project.

Basic Concept - We need to find the *Critical Path* (the set of activities which, if delayed, will cause completion of the entire project to be delayed?).

This is the longest path (sequence of connected activities that lead from the starting node to the completion node) in the project network - the time it takes to complete the tasks on the critical path is the total time required to complete the project.

Note that all paths must be traversed to complete the project - the critical path is simply the longest in the network.

PERT (Project Evaluation Review Technique) - Developed in late 1950's for planning, scheduling, and controlling the Polaris missile project. Includes uncertainties (and thus probabilistic) activity times. Activities are often represented on the arcs in a PERT Network display.

CPM (Critical Path Method) - Developed independently by DuPont for scheduling and controlling industrial projects with deterministic activity times. Activities are often represented on the nodes in a CPM Network display.

The distinction between the separate methods (PERT & CPM) has largely disappeared and they are now considered together.

These techniques can be used to answer questions such as:

- What is the expected completion date of the project?**
- What are the scheduled start and completion dates for each specific activity?**
- Which activities are "critical" in the sense that if the activity is delayed, the entire project is delayed?**
- How long can "noncritical" activities be delayed before they cause a delay in the total project?**

A. Deterministic PERT/CPM

Activities - individual tasks that collectively comprise a project.

PERT Network - graphical chronological representation of the interrelationships between the activities that comprise a project.

Node - representation of when an event (point at which an activity starts) occurs.

Arc - representation of the length of time to complete the corresponding activity

The steps in the analysis of a PERT Network:

Step #1 Identify the

- individual activities
- immediate predecessor(s) for each activity
- completion times for each activity

Step #2 Build a graphical representation of the project network

- identify the nodes (start/completion points of activities)
- connect node with directed arcs (lines that represent activities)

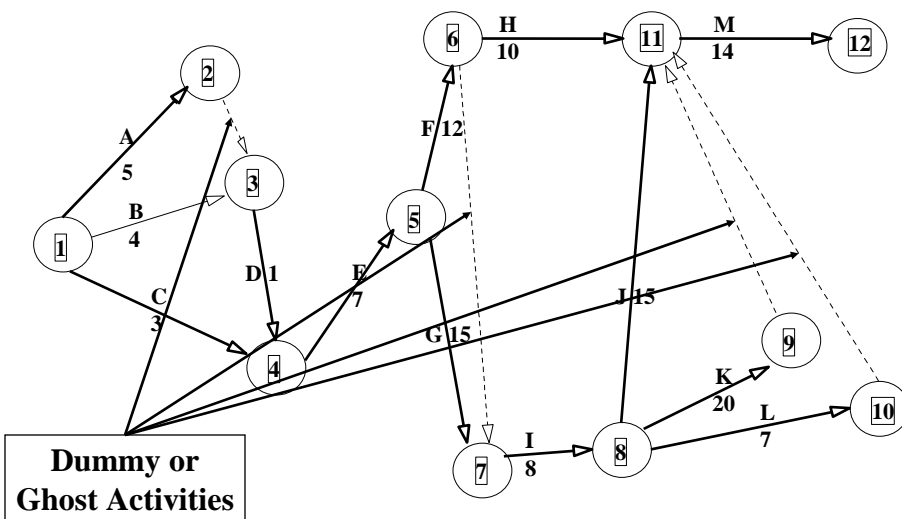
Step #3 Identify the critical path

- Identify the earliest start (ES) and earliest finish (EF) times for each node. Note that the earliest start time for the first node (or any node with no predecessor) is zero, and the earliest start time for any other node is the *largest of the earliest finish times for all predecessor nodes*.
- Identify the latest start (LS) and latest finish (LF) times for each node. Note that the latest finish time for the any node is the *smallest of the latest start times for all activities leaving the node*.
- Identify the path through which $ES=LS$ (or $EF=LF$), i.e., the slack is zero (why?). This is the critical path.
- If two nodes are connected but do not have an activity that occurs between them, use a dummy activity and assign it a time of zero, then proceed as usual.

Example - Use deterministic PERT/CPM to find the critical path, slack times, and time to completion for the following project.

Activity	Completion Time	Immediate Predecessor(s)
A	5	---
B	4	---
C	3	---
D	1	AB
E	7	CD
F	12	E
G	15	E
H	10	F
I	8	FG
J	15	I
K	20	I
L	7	I
M	14	HJKL

A graphical representation of the project network could look like this:



Note that this problem is analogous to the Shortest Route (Path) Problem - the LP formulation is

$$\text{maximize } C = \sum_{i=0}^N \sum_{j=0}^N c_{ij} x_{ij}$$

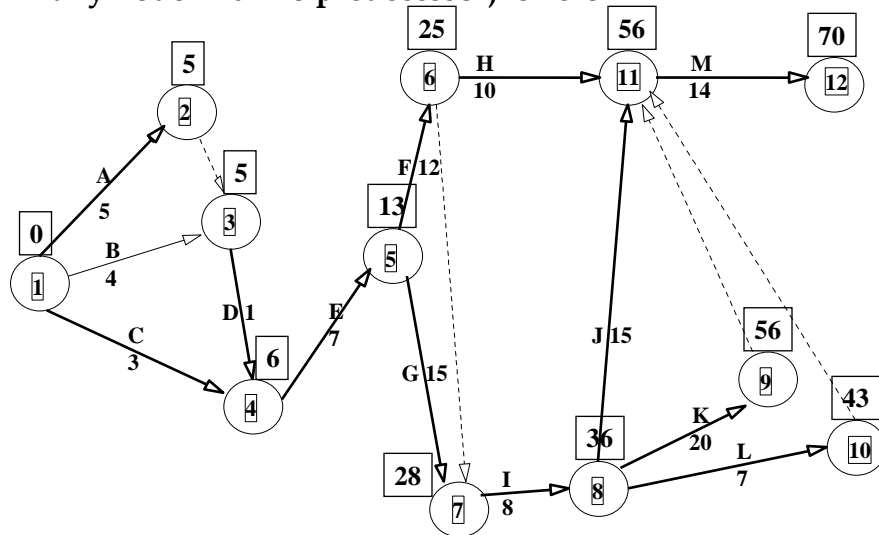
$$\text{subject to } \sum_{i=0}^N x_{ik} = \sum_{j=0}^N x_{kj} \quad (\text{for intermediate nodes } k=1, \dots, N-1)$$

$$\sum_{j=0}^N x_{0j} = 1 \quad (\text{requirement to start the project})$$

$$\sum_{i=0}^N x_{iN} = 1 \quad (\text{requirement to complete the project})$$

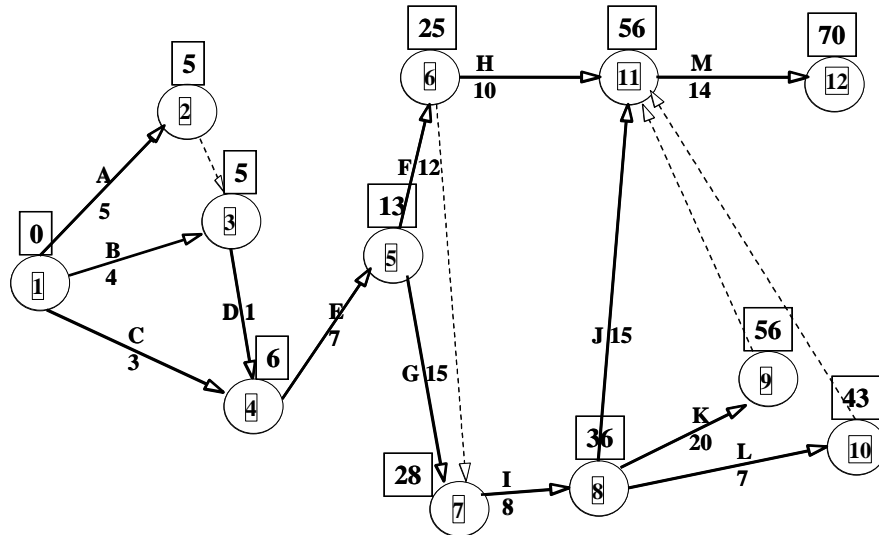
All $x_{ij} \geq 0$ (nonnegativity)

Identify the *Earliest Possible Event (start) Times* or TE times for each node. Note that the TE for the first node (or any node with no predecessor) is zero



and the TE for any other node is the *largest of the Earliest Possible Finish Times for all predecessor nodes.*

The *Critical Path* is the longest path that spans the network:



or A-D-E-G-I-K-M at a time to completion of 70 days.

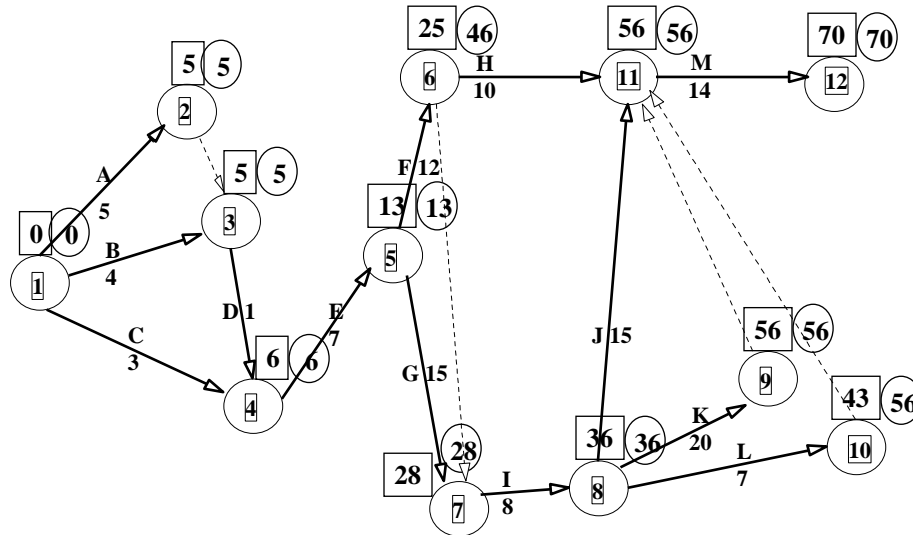
Although we can conceivably find the *Critical Path* by identifying the longest path that spans the network, this can be very difficult in large, complex projects.

We could also use information on how much leeway we have in when we must start various tasks.

Let's determine the *Latest Allowable Event (start) Times* or TLs times for each node.

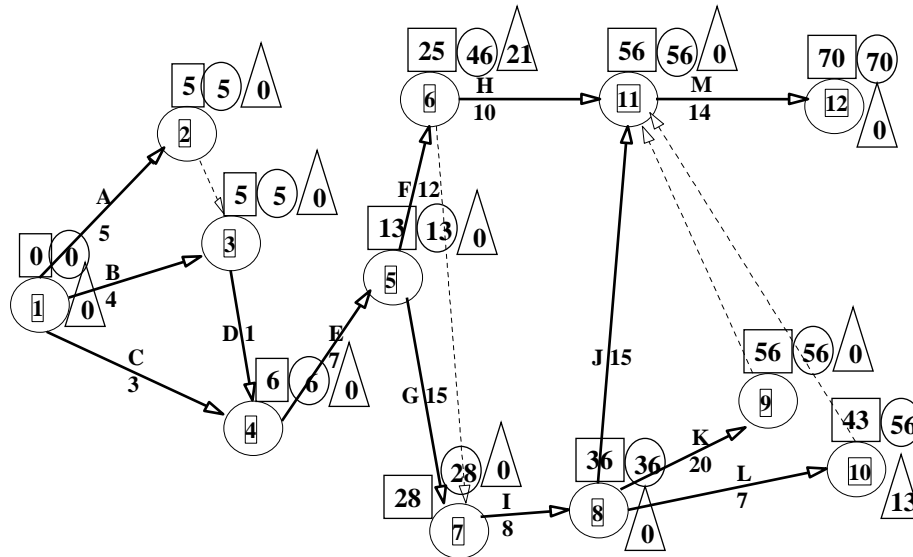
TLs are computed on a backward pass through the network, where activity completion times are subtracted.

Here is a graphical display of the TLs:



What do you notice about the *Critical Path*?

The differences between TEs and TLs are zero on the Critical Path! Is there an economic interpretation for these values?



They represent the leeway in when we must start a task.

Why? - Because these values (called *Event Slack Times* - not *SLACK VARIABLES*) are the difference between TEs (the earliest point at which we can start a task) and TLs (the latest earliest point at which we can start a task). Now think again about what it means to be on a *critical path*.

Critical Path - the series of tasks in a project for which the Event Slack Times are zero (i.e., we can have NO SLIPPAGE on start times of these tasks if we are to complete the project in the minimum number of days).

This is what makes these events **critical!**

Note that we can derive still more useful information from our network

We can easily find the Earliest Finishing Times (EF) and Latest Finishing Times (LF) - if t is the time to finish an activity, then $EF = TE + t$ and $LF = TL + t$

Activity	TE	TL	EF	LF	Slack	Critical?
A	0	0	5	5	0	yes
B	0	1	4	5	1	no
C	0	3	3	6	3	no
D	5	5	6	6	0	yes
E	6	6	13	13	0	yes
F	13	16	25	28	3	no
G	13	13	28	28	0	yes
H	25	46	35	56	21	no
I	28	28	36	36	0	yes
J	36	41	51	56	5	no
K	36	49	43	56	13	no
L	36	36	56	56	0	yes
M	56	56	70	70	0	yes

Crashing - If the likelihood of project completion in a given time is unacceptable, or if management wishes to add resources (perhaps more workers or overtime) to decrease activity times, we need to consider *crashing*. In other words, we want to identify the activities which we will benefit most by when adding resources. The steps of crashing are:

Step 1 Estimate the normal activity cost (C_n) and the cost under maximum crashing (C_c).

Step 2 Estimate the time under maximum crashing (t^1)

Step 3 Estimate the activity cost under maximum crashing (k), i.e.,

$$k = \frac{C_c - C_n}{t - t^1}$$

Example: Consider this data for our previous example...

Activity	Completion		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F	12.00	3.00	\$110	\$200	\$10.00
G*	15.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	8.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	14.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

One can use the information on *crashing* to:

- select tasks to crash (shorten) if the critical path is unacceptably long;
- construct a Time-Cost Trade-Off Curve

Crash the cheapest critical activity by the maximum amount and record the resulting total i) time and ii) cost to complete the project - note that crashing will broaden the number of activities on the critical path;

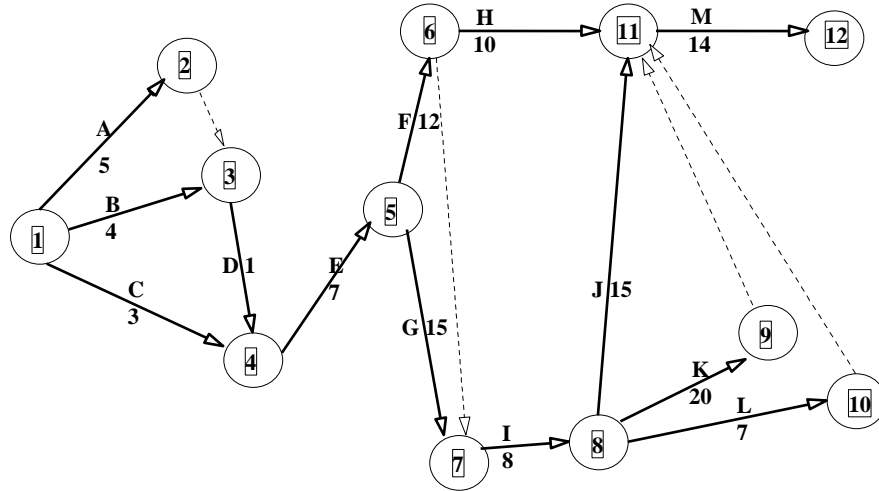
Crash the cheapest critical activity by the maximum amount and record the resulting total i) time and ii) cost to complete the project;

Repeat until sufficient data are collected.

For our example the cheapest critical activity to crash is G:

Activity	Completion		Costs		Crash Cost
	Time		Normal	Crash	Per Day
	Normal	Crash			
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F	12.00	3.00	\$110	\$200	\$10.00
G*	15.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	8.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	14.00	7.00	\$1350	\$1175	\$25.00
			<u>\$4505</u>		

Although we are allowed to crash up to seven days out of the time to complete Activity G, we are restricted to crashing only three days at this point (why?)

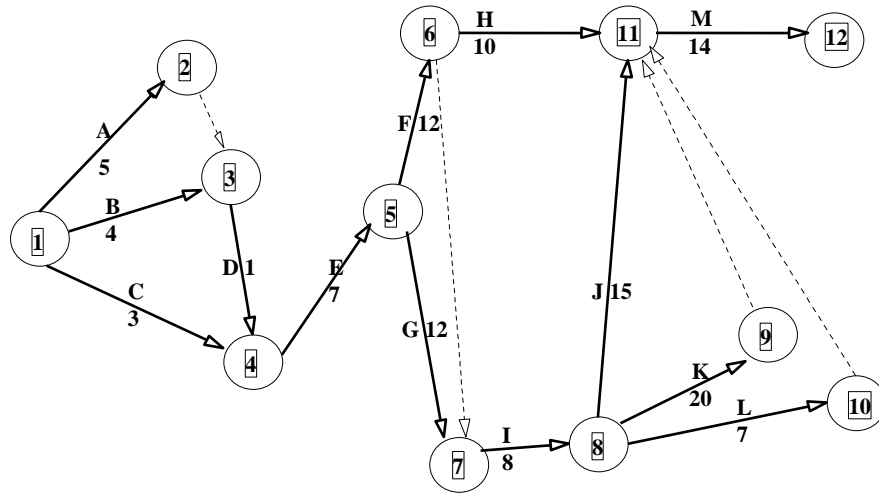


Here's what happens when we crash is G:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
G	3.00	\$5.00	67	\$4520.00

We now go looking for the cheapest remaining critical activity to crash (keeping in mind that Activity F is now critical):

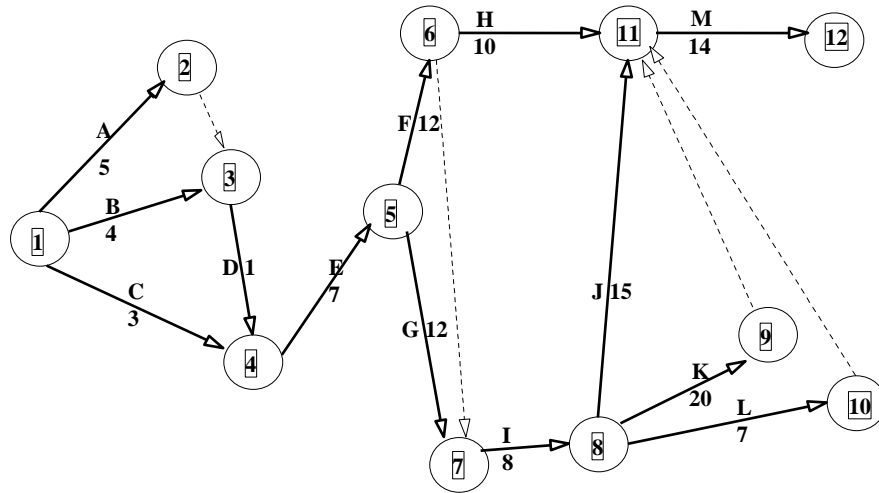
The network flow diagram revised to reflect the crashing of three days out of the time to complete Activity G:



The cheapest critical activity to crash is F at \$15.00 (why?):

Activity	Completion Time		Costs		Crash Cost Per Day
	Normal	Crash	Normal	Crash	
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F*	12.00	3.00	\$110	\$200	\$10.00
G*	12.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	8.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	14.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

Although crashing F costs only \$10.00 per day, we will also have to crash G one day for every day we crash F!

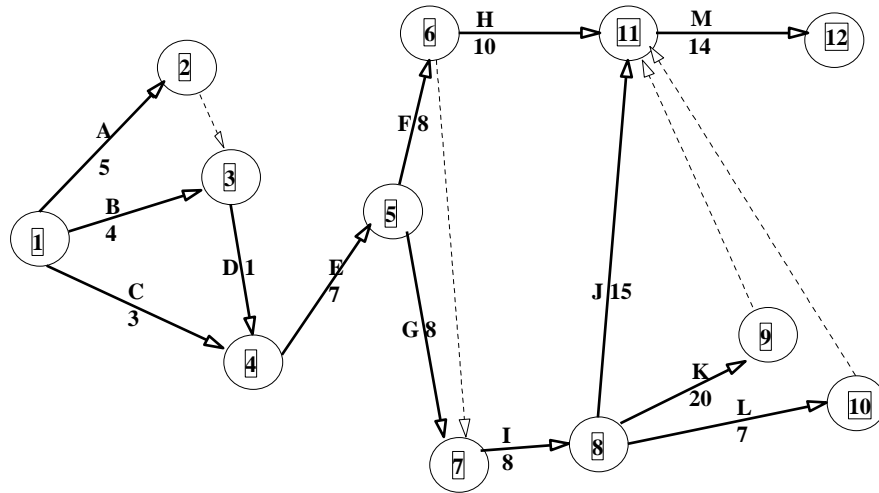


Here's what happens when we crash F:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4580.00

We resume our search for the cheapest remaining critical activity to crash (keeping in mind that we can not crash Activity F or Activity G any further):

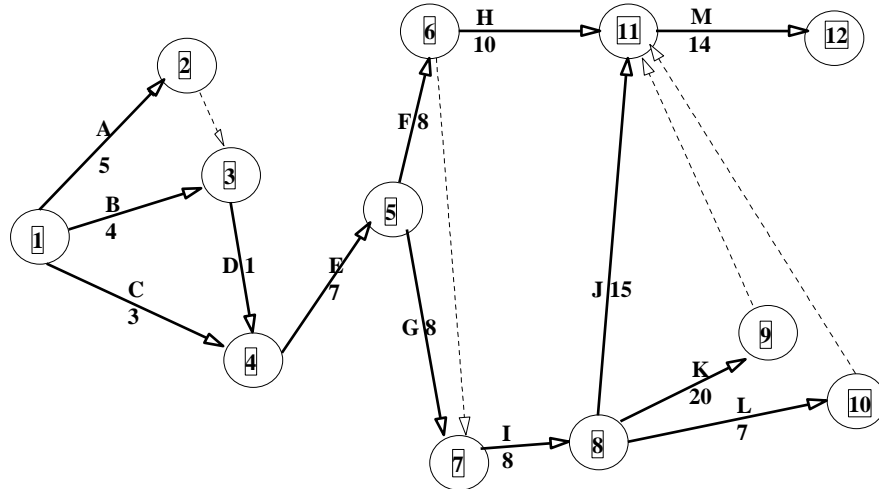
The network flow diagram revised to reflect the crashing of four days out of the time to complete Activity F (and G):



The cheapest critical activity to crash is I at \$20.00 (why?):

Activity	Completion Time		Costs		Crash Cost Per Day
	Normal	Crash	Normal	Crash	
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	8.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	14.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We are capable of crashing I by 4.5 days at \$20.00 per day, and we could actually crash up to 8 days out of the time to complete I before other task(s) become critical (why?)



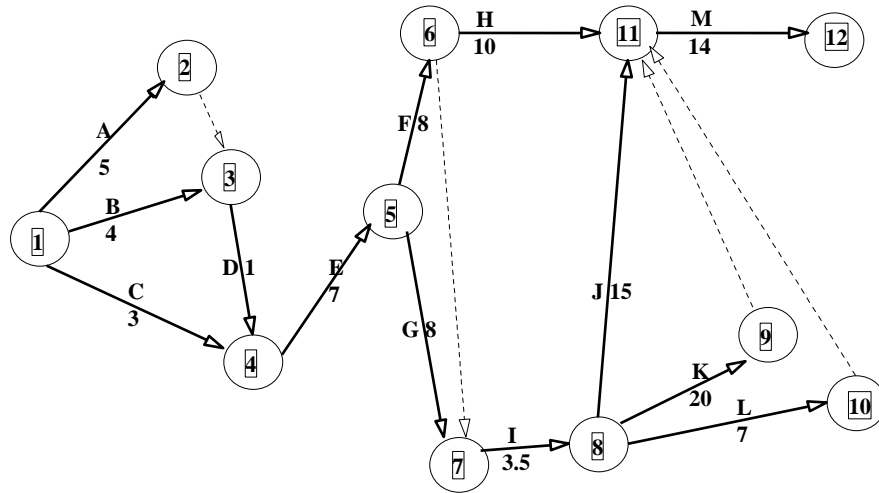
If we crashed I by 8 days, GIK takes $8+0+15=23$ days and F&H remain noncritical precedents to M at $8+10=18$ days.

Here's what happens when we crash I:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4600.00
3. I	4.50	\$20.00	58.5	\$4690.00

We resume our search for the cheapest remaining critical activity to crash:

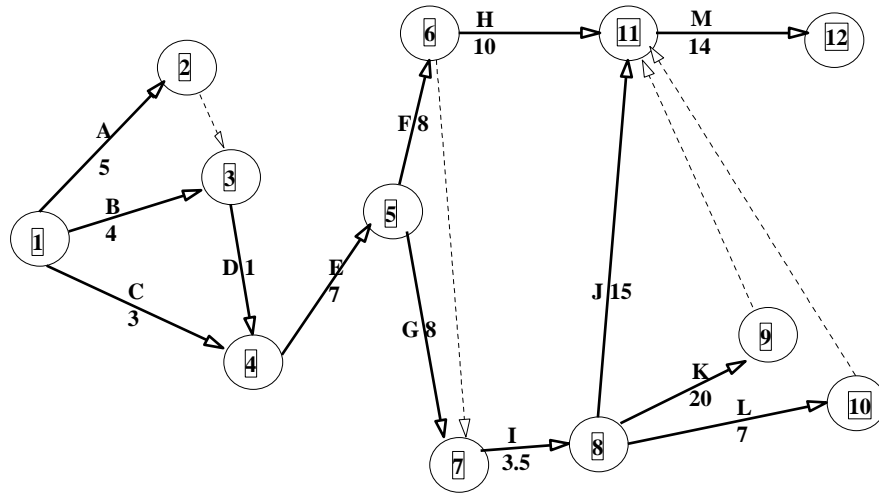
The network flow diagram revised to reflect the crashing of four and 1/2 days out of the time to complete Activity I:



The cheapest critical activity to crash is M at \$25.00 (why?):

Activity	Completion		Costs		Crash Cost Per Day
	Normal	Crash	Normal	Crash	
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	5.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	14.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can crash M by the allowed 7 days at \$25.00 per day (why?)

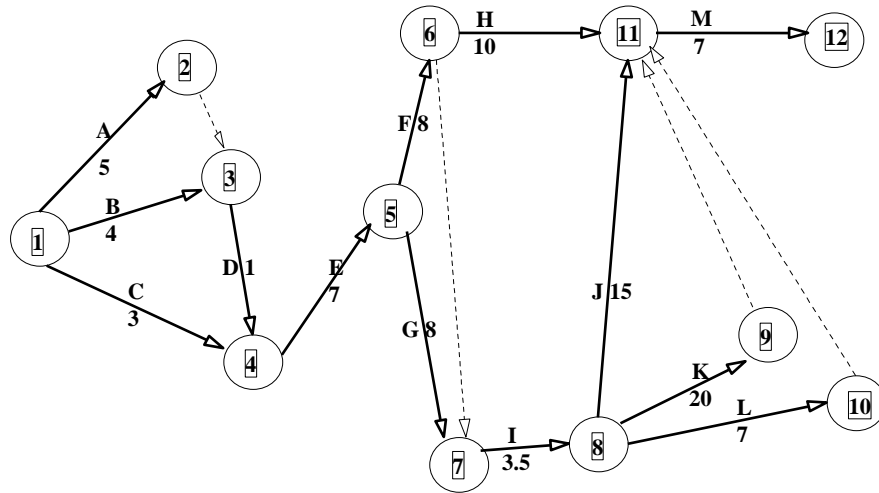


Here's what happens when we crash M:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4600.00
3. I	3.00	\$20.00	60	\$4660.00
4. M	7.00	\$25.00	53	\$4835.00

We resume our search for the cheapest remaining critical activity to crash:

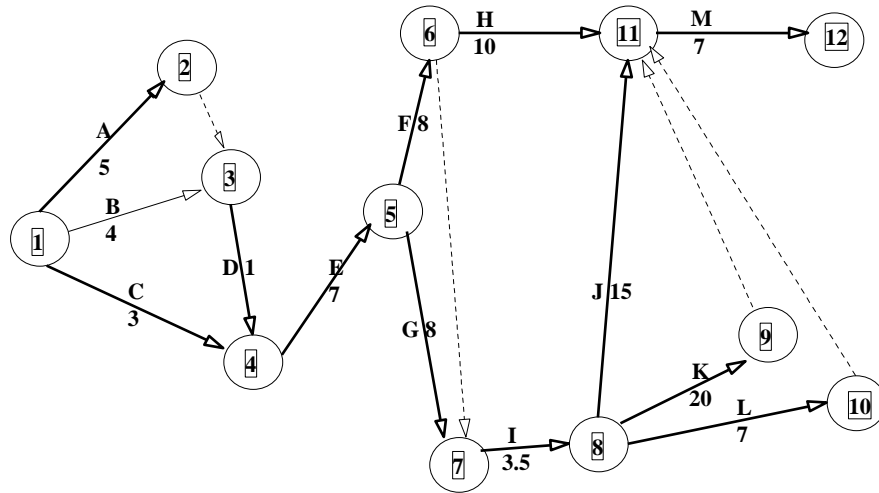
The network flow diagram revised to reflect the crashing of seven days out of the time to complete Activity M:



The cheapest critical activity to crash is E at \$32.50 (why?):

Activity	Completion Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	5.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can crash E by the allowed 2 days at \$32.50 per day (why?)

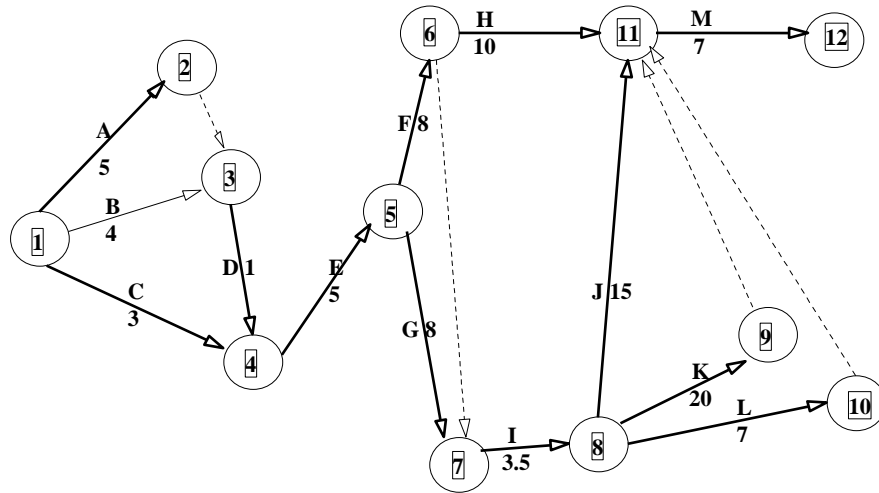


Here's what happens when we crash E:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4600.00
3. I	3.00	\$20.00	60	\$4660.00
4. M	7.00	\$25.00	53	\$4835.00
5. E	2.00	\$32.50	51	\$4900.00

We resume our search for the cheapest remaining critical activity to crash (keeping in mind that we can crash no further on Activity E):

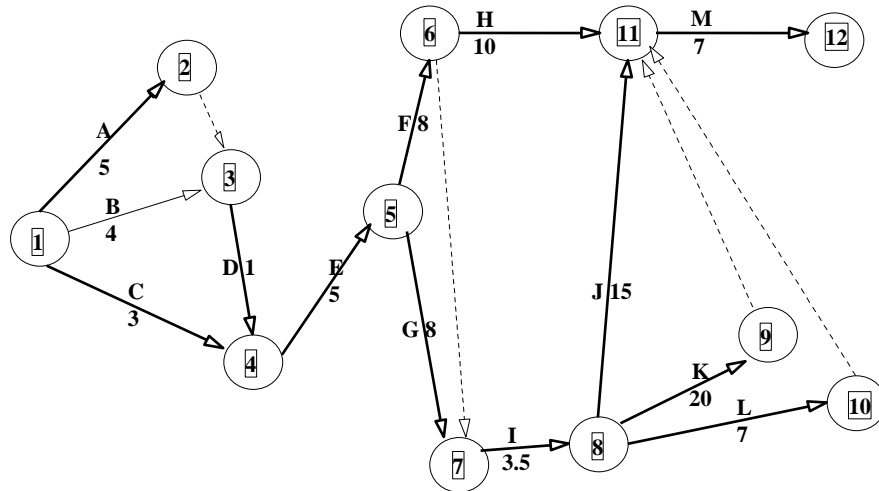
The network flow diagram revised to reflect the crashing of two days out of the time to complete Activity E:



The cheapest critical activity to crash is K at \$50.00 (why?):

Activity	Completion Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	7.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	5.00	3.50	\$160	\$250	\$20.00
J	15.00	9.00	\$500	\$650	\$25.00
K*	20.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We are allowed to crash K by up to 12 days at \$50.00 per day but are only capable of crashing K by 5 days (why?)



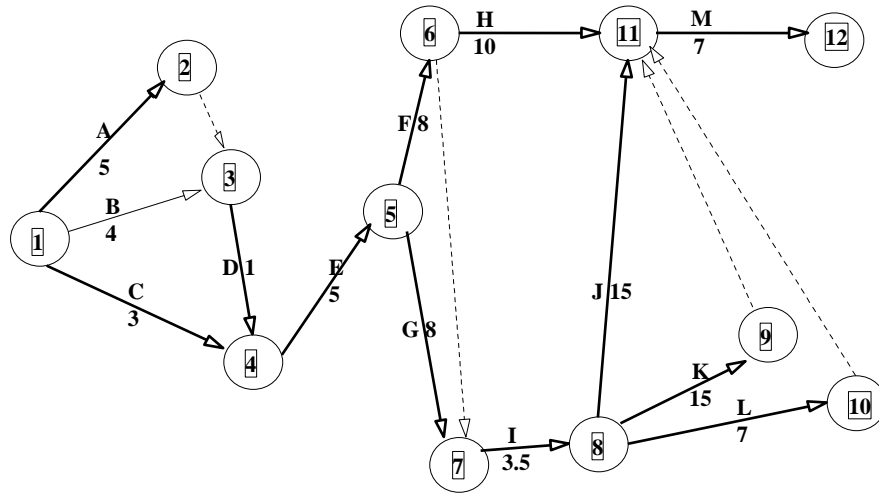
Once we crash K by 5 days, J becomes critical!

Here's what happens when we crash K:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4600.00
3. I	3.00	\$20.00	60	\$4660.00
4. M	7.00	\$25.00	53	\$4835.00
5. E	2.00	\$32.50	51	\$4900.00
6. K	5.00	\$50.00	47	\$5150.00

We resume our search for the cheapest remaining critical activity to crash (keeping in mind that we can crash no further on Activity K without also crashing on Activity J):

The network flow diagram revised to reflect the crashing of five days out of the time to complete Activity K:



We can crash 1 day out of Activity A at \$75.00/day:

Completion

Activity	Time		Costs		Crash Cost Per Day
	Normal	Crash	Normal	Crash	
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	15.00	9.00	\$500	\$650	\$25.00
K*	15.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can crash $\frac{1}{4}$ day out of Activity D at \$300.00/day:
Completion

Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
[]					
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	15.00	9.00	\$500	\$650	\$25.00
K*	15.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can crash 6 days out of Activity J&K at \$75.00/day:
Completion

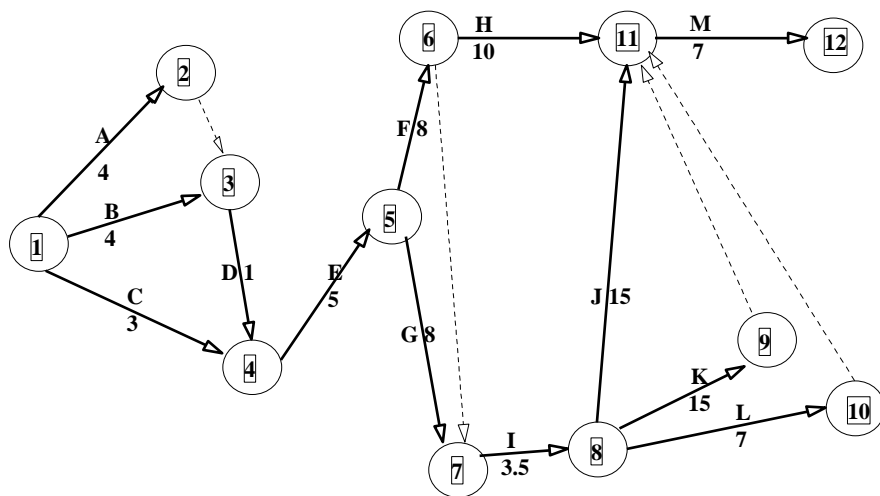
Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	5.00	3.00	\$150	\$300	\$75.00
B	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
[]					
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can arbitrarily choose to crash either Activity A or Activities J&K (guess what we will crash on the next iteration?) - we'll arbitrarily choose to crash on Activity A:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4600.00
3. I	3.00	\$20.00	60	\$4660.00
4. M	7.00	\$25.00	53	\$4835.00
5. E	2.00	\$32.50	51	\$4900.00
6. K	5.00	\$50.00	47	\$5150.00
7. A	1.00	\$75.00	46	\$5225.00

We resume our search for the cheapest remaining critical activity to crash (keeping in mind that we can crash on Activities J & K for the same cost as Activity A in this iteration):

The network flow diagram revised to reflect the crashing of one day out of the time to complete Activity A:



We can crash one day from Activities A&B at \$112.50/day:
Completion

Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	15.00	9.00	\$500	\$650	\$25.00
K*	15.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can crash 1/4 day from Activity D at \$300.00/day:
Completion

Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	4.00	3.00	\$150	\$300	\$75.00
B*	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35	---	---
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	15.00	9.00	\$500	\$650	\$25.00
K*	15.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

..or we now crash 6 days out of Activity J&K at \$75.00/day:

Completion

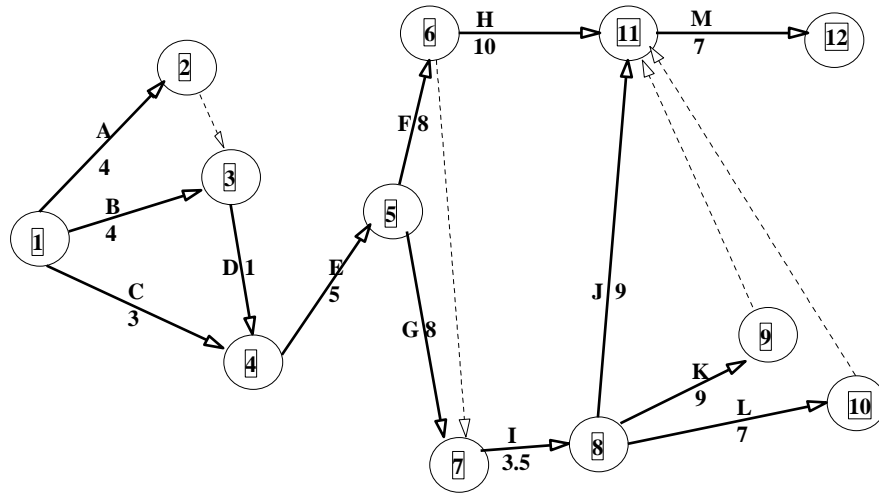
Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	4.00	3.00	\$150	\$300	\$75.00
B*	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

Here's what happens when we crash Activities J&K by six days:

Crashed Activity	Time Saved	Crash Cost	New Project Completion	
			Time	Cost
1. G	3.00	\$5.00	67	\$4520.00
2. F	4.00	\$10.00	63	\$4560.00
G	4.00	\$5.00	63	\$4600.00
3. I	3.00	\$20.00	60	\$4660.00
4. M	7.00	\$25.00	53	\$4835.00
5. E	2.00	\$32.50	51	\$4900.00
6. K	5.00	\$50.00	47	\$5150.00
7. A	1.00	\$75.00	46	\$5225.00
8. J	6.00	\$150.00	40	\$5375.00
9. K	6.00	\$300.00	40	\$5675.00

We resume our search for the cheapest remaining critical activity to crash.

The network flow diagram revised to reflect the crashing of six days out of the time to complete Activities J & K:



We can crash one day from Activities A&B at \$112.50/day:
Completion

Activity	Time		Costs		Crash Cost Per Day
	Normal	Crash	Normal	Crash	
C	3.00	3.00	\$35		
D*	1.00	0.75	\$100	\$175	\$300.00
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	9.00	9.00	\$500	\$650	\$25.00
K*	9.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

We can crash $\frac{1}{4}$ day from Activity D at \$300.00/day:

Completion

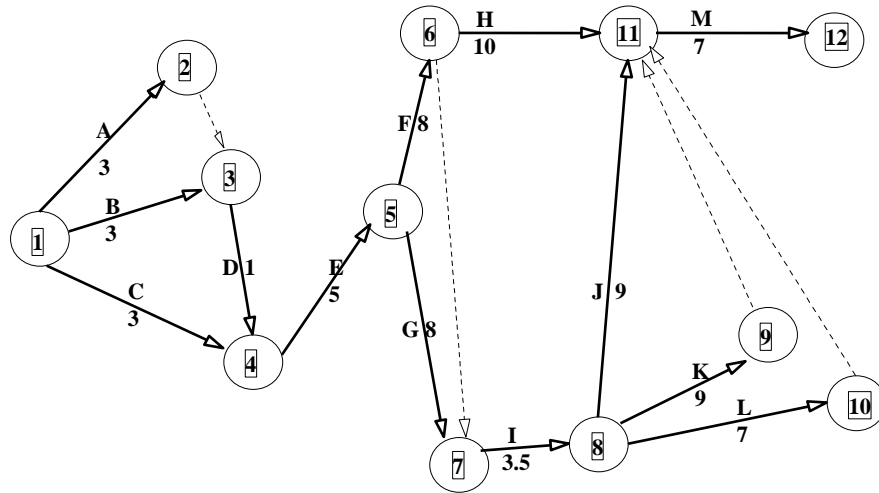
Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	4.00	3.00	\$150	\$300	\$75.00
B*	4.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	9.00	9.00	\$500	\$650	\$25.00
K*	9.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

Here's what happens when we crash Activities A&B by one day:

	Crashed Activity	Time Saved	Crash Cost	New Project Completion	
				Time	Cost
1.	G	3.00	\$5.00	67	\$4520.00
2.	F	4.00	\$10.00	63	\$4560.00
	G	4.00	\$5.00	63	\$4600.00
3.	I	3.00	\$20.00	60	\$4660.00
4.	M	7.00	\$25.00	53	\$4835.00
5.	E	2.00	\$32.50	51	\$4900.00
6.	K	5.00	\$50.00	47	\$5150.00
7.	A	1.00	\$75.00	46	\$5225.00
8.	J	6.00	\$150.00	40	\$5375.00
9.	K	6.00	\$300.00	40	\$5675.00
10.	A	1.00	\$75.00	39	\$5750.00
11.	B	1.00	\$37.50	39	\$5787.50

We resume our search for the cheapest remaining critical activity to crash.

The network flow diagram revised to reflect the crashing of one day out of the time to complete Activities A & B:



We can crash $\frac{1}{4}$ day from Activity D at \$300.00/day:
Completion

Activity	Time		Costs		Crash Cost
	Normal	Crash	Normal	Crash	Per Day
A*	3.00	3.00	\$150	\$300	\$75.00
B*	3.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	9.00	9.00	\$500	\$650	\$25.00
K*	9.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

Here's what happens when we crash Activity D by 1/4 day:

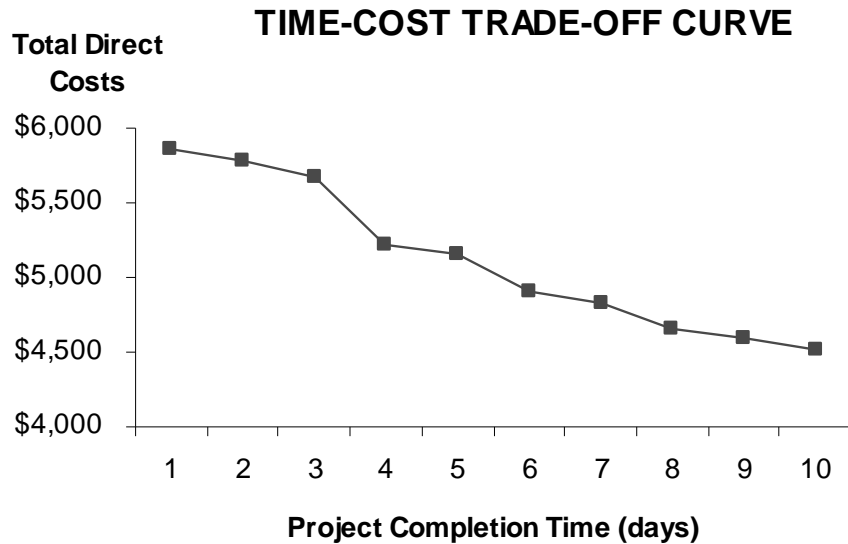
	Crashed Activity	Time Saved	Crash Cost	New Project Completion	
				Time	Cost
1.	G	3.00	\$5.00	67	\$4520.00
2.	F	4.00	\$10.00	63	\$4560.00
	G	4.00	\$5.00	63	\$4600.00
3.	I	3.00	\$20.00	60	\$4660.00
4.	M	7.00	\$25.00	53	\$4835.00
5.	E	2.00	\$32.50	51	\$4900.00
6.	K	5.00	\$50.00	47	\$5150.00
7.	A	1.00	\$75.00	46	\$5225.00
8.	J	6.00	\$150.00	40	\$5375.00
9.	K	6.00	\$300.00	40	\$5675.00
10.	A	1.00	\$75.00	39	\$5750.00
11.	B	1.00	\$37.50	39	\$5787.50
12.	D	0.25	\$75.00	38.75	\$5862.60

We resume our search for the cheapest remaining critical activity to crash.

We cannot crash any other Activity:

Activity	Completion		Costs		Crash Cost Per Day
	Time				
	Normal	Crash	Normal	Crash	
A*	3.00	3.00	\$150	\$300	\$75.00
B*	3.00	2.00	\$75	\$150	\$37.50
C	3.00	3.00	\$35		
D*	0.75	0.75	\$100	\$175	\$300.00
E*	5.00	5.00	\$95	\$160	\$32.50
F*	8.00	3.00	\$110	\$200	\$10.00
G*	8.00	8.00	\$40	\$75	\$5.00
H	10.00	6.00	\$100	\$150	\$12.50
I*	3.50	3.50	\$160	\$250	\$20.00
J*	9.00	9.00	\$500	\$650	\$25.00
K*	9.00	8.00	\$300	\$900	\$50.00
L	7.00	2.00	\$1500	\$1000	\$100.00
M*	7.00	7.00	\$1350	\$1175	\$25.00
			\$4505		

At some point we exhaust all possible crashes. We then construct our Time-Cost Trade-Off Curve



B. Probabilistic (Stochastic) PERT/CPM

If there is uncertain about the completion time of some or all activities, we follow the same steps as in Deterministic PERT/CPM except:

in Step #1 we identify the most optimistic, most likely, and most pessimistic completion times for each, and

in Step #2 we calculate the expected completion time for each activity and the variance of the completion time for each activity. to do these calculations we:

Let a = Optimistic time estimate
 m = Most likely time estimate
 b = Pessimistic time estimate

for an individual activity.

Then under the assumption that the uncertain activity times are best described by the *Modified Beta* probability distribution:

$$t = \frac{a + 4m + b}{6} \quad s^2 = \frac{(b - a)^2}{6}$$

where t = expected activity completion time
 s^2 = variance of activity completion time

We can use this information to solve these PERT problems. Recognize that:

The expected value (or mean) of the completion time for the project is the sum of the expected values for the activities on the critical path.

The variance (or squared standard deviation) of the completion time for the project is the sum of the variances for the activities on the critical path.

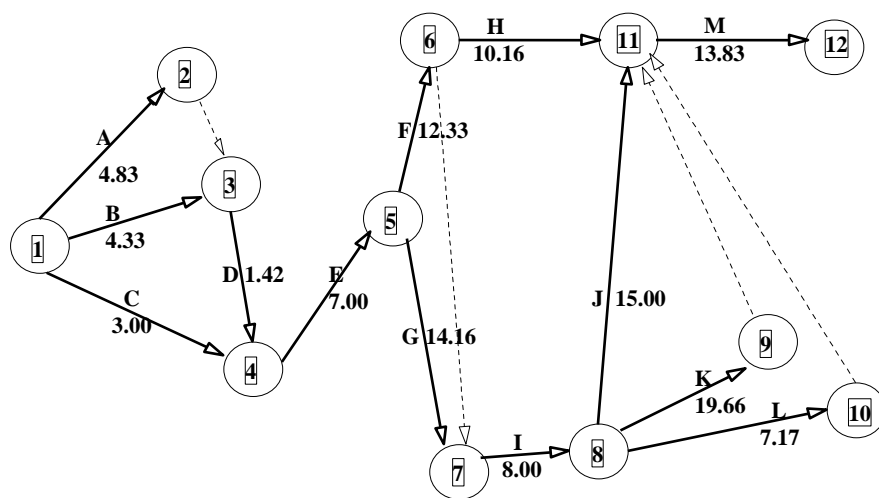
Suppose we have this additional information for our previous example...

Activity	Estimated Completion Times			Expected	
	Optimistic (a)	Most Likely (m)	Pessimistic(b)	Time	Variance
A*	3	5	6		
B	2	4	8		
C	1	3	5		
D*	1	1	4		
E*	5	7	9		
F	8	12	18		
G*	8	15	17		
H	8	10	13		
I*	6	8	10		
J	15	15	15		
K*	16	20	22		
L	6	7	9		
M*	11	14	16		

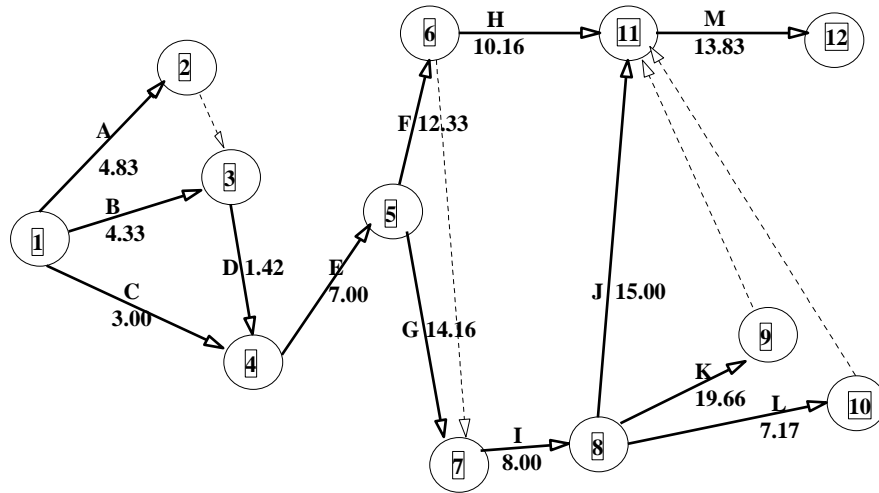
We can use these values (a, m, and b) to find the expected value and variance for each activity. Then we can add the values corresponding to the critical path together to develop estimates for this path.

Activity	Estimated Completion Times			Expected	
	Optimistic (a)	Most Likely (m)	Pessimistic(b)	Time	Variance
A*	3	5	6	4.83	0.25
B	2	4	8	4.33	1.00
C	1	3	5	3.00	0.44
D*	1	1	4	1.42	0.34
E*	5	7	9	7.00	0.44
F	8	12	18	12.33	2.78
G*	8	15	17	14.16	2.25
H	8	10	13	10.16	0.69
I*	6	8	10	8.00	0.44
J	15	15	15	15.00	0.00
K*	16	20	22	19.66	1.00
L	6	7	9	7.17	0.25
M*	11	14	16	13.83	0.69

The network flow diagram revised to reflect the crashing of one day out of the time to complete Activities A & B:



The network flow diagram revised to reflect the crashing of one day out of the time to complete Activities A & B:



Now we have estimates for the critical path:

$$\mu_T = 4.83 + 1.42 + 7.00 + 14.16 + 8.00 + 19.66 + 13.83 = 68.94$$

$$\sigma_T^2 = 0.25 + 0.34 + 0.44 + 2.25 + 0.44 + 1.00 + 0.69 = 5.42$$

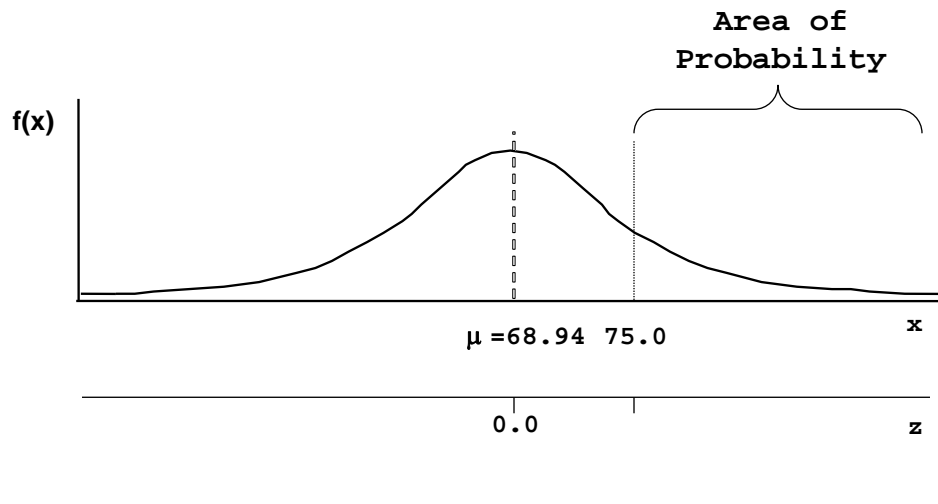
$$\Rightarrow \sigma_T = 2.33$$

Note that we can only add the variances for various tasks if the times to complete those tasks are *statistically independent*.

A form of the *Central Limit Theorem* provides that T (the time to complete all activities on the critical path) is approximately normally distributed.

Now we could answer questions such as *What is the probability that completion of the project will take at least 75 days?*

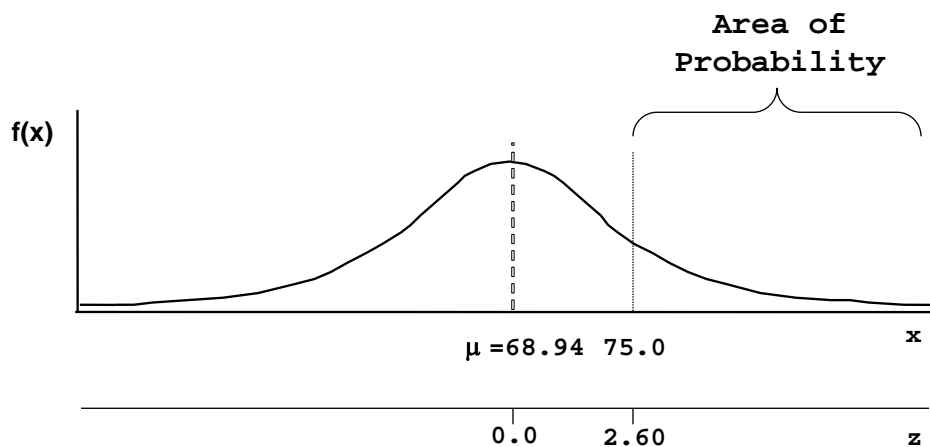
Example: for a normal random variable x (number of days to project completion) with a mean of 68.94 and a standard deviation of 2.33, what is the probability that x is at least 75.0?



Using the z -transformation (standard normal), we can restate the problem in the following manner:

$$\begin{aligned}
 P(75.0 \leq x) &= P\left(68.94 \leq \frac{x - \mu}{\sigma} \leq \infty\right) - P\left(68.94 \leq \frac{x - \mu}{\sigma} \leq 75.00\right) \\
 &= 0.5000 - P\left(\frac{68.94 - 68.94}{2.33} \leq \frac{x - \mu}{\sigma} \leq \frac{75.0 - 68.94}{2.33}\right) \\
 &= 0.5000 - P(0.0 \leq z \leq 2.60)
 \end{aligned}$$

By overlaying the z-axis on our original graphical depiction of this problem, we see how the z-transformation works:



Areas Under the Standard Normal Distribution (Appendix, Table A)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Then use the result from standard normal probability table to find the ultimate answer:

$$P(0.0 \leq z \leq 2.60) = 0.4953$$

$$P(2.60 \leq z) = 0.5000 - 0.4953 = 0.0047$$