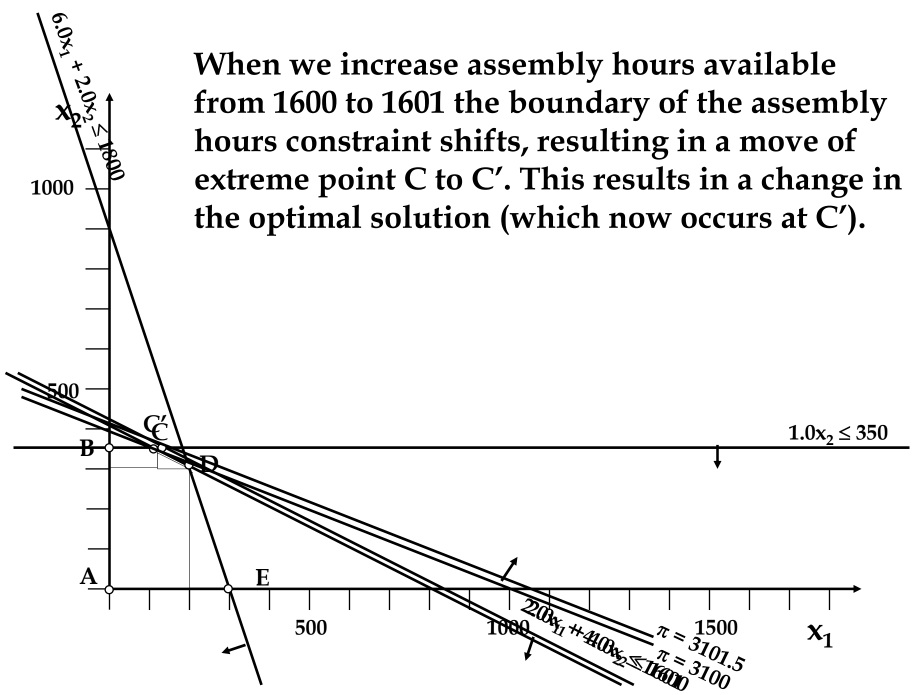
D. Duality, Sensitivity, & the Simplex Method

1. **Shadow Price** - ***net change*** **in the objective function that would result from a one unit increase in the right-hand side of a particular constraint.**

2. **Opportunity Cost** - ***improvement*** **in the objective function that would result from a one unit increase in the right-hand side of a particular constraint.**

For example, suppose we actually had 1601 assembly hours available (instead of 1600) in the Television Production Problem, i.e.,

$$2.0x_1 + 4.0x_2 \leq 1601 \text{ (\# of assembly hours available)}$$



When we increase assembly hours available from 1600 to 1601 the boundary of the assembly hours constraint shifts, resulting in a move of extreme point C to C′. This results in a change in the optimal solution (which now occurs at C′).

We can find the new optimal solution by first finding the values of the decision variables by simultaneously solving for the constraint boundary lines that intersect at the optimal point (C′)

$$\begin{array}{rl} & 2.0x_1 + 4.0x_2 = 1601 \\ -4(& \quad\quad 1.0x_2 = \ 350) \\ \hline & 2.0x_1 + 0.0x_2 = \ 201 \rightarrow x_1 = 100.5 \end{array}$$

which implies that

$$2.0x_1 + 4.0x_2 = 2.0(100.5) + 4.0x_2 = 1601 \rightarrow 4.0x_2 = 1400 \rightarrow x_2 = 350$$

Substitution of the optimal values of the decision variables (found in the previous step) into the objective function leads to the optimal solution:

$$\pi = 3.0x_1 + 8.0x_2 = 3.0(100.5) + 8.0(350) = 3101.5$$

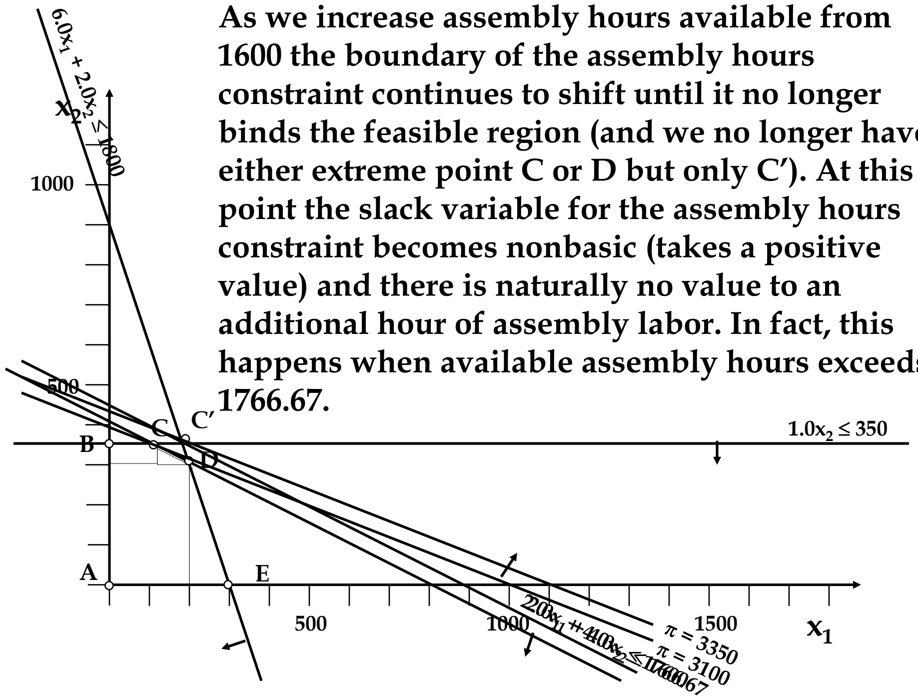
So the new optimal solution is:

Produce 100.5 black & white sets, 350 color sets, and earn $3101.5 profit

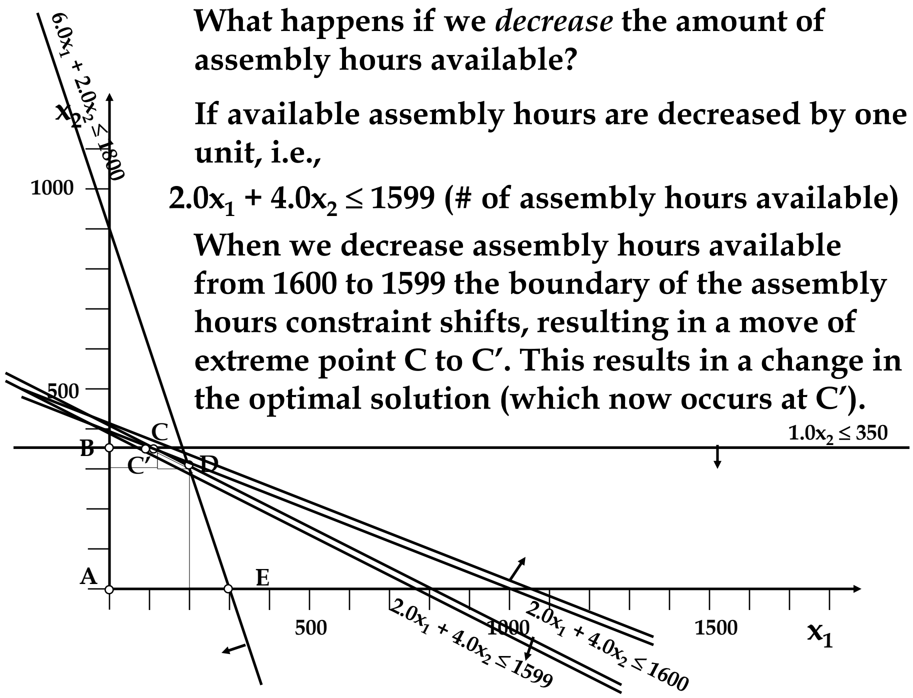
This suggests that the shadow price for one unit of assembly labor is $3101.5 - $3100 = $1.50 WITH RESPECT TO THE ORIGINAL FORMULATION!

Note that:

- **the shadow price is $1.50 because an additional assembly hour allows us to make 1/2 additional black and white television set (and we earn $3.00 per set)**
- **because this is a *maximization* problem, the shadow price and opportunity cost *are equivalent* (why?)**
- **this shadow price is only effective over a limited range of values for available assembly labor hours (why?)**



As we increase assembly hours available from 1600 the boundary of the assembly hours constraint continues to shift until it no longer binds the feasible region (and we no longer have either extreme point C or D but only C′). At this point the slack variable for the assembly hours constraint becomes nonbasic (takes a positive value) and there is naturally no value to an additional hour of assembly labor. In fact, this happens when available assembly hours exceeds 1766.67.



What happens if we *decrease* the amount of assembly hours available?

If available assembly hours are decreased by one unit, i.e.,

$$2.0x_1 + 4.0x_2 \leq 1599 \text{ (\# of assembly hours available)}$$

When we decrease assembly hours available from 1600 to 1599 the boundary of the assembly hours constraint shifts, resulting in a move of extreme point C to C′. This results in a change in the optimal solution (which now occurs at C′).

We can again find the new optimal solution by first finding the values of the decision variables by simultaneously solving for the constraint boundary lines that intersect at the optimal point (C′)

$$
\begin{array}{rl}
 & 2.0x_1 + 4.0x_2 = 1599 \\
-4(& \quad\quad 1.0x_2 = 350) \\
\hline
 & 2.0x_1 + 0.0x_2 = 199 \rightarrow x_1 = 99.5
\end{array}
$$

which implies that

$$2.0x_1 + 4.0x_2 = 2.0(99.5) + 4.0x_2 = 1599 \rightarrow 4.0x_2 = 1400 \rightarrow x_2 = 350$$

Substitution of the optimal values of the decision variables (found in the previous step) into the objective function leads to the optimal solution:

$$\pi = 3.0x_1 + 8.0x_2 = 3.0(99.5) + 8.0(350) = 3098.5$$

So the new optimal solution is:

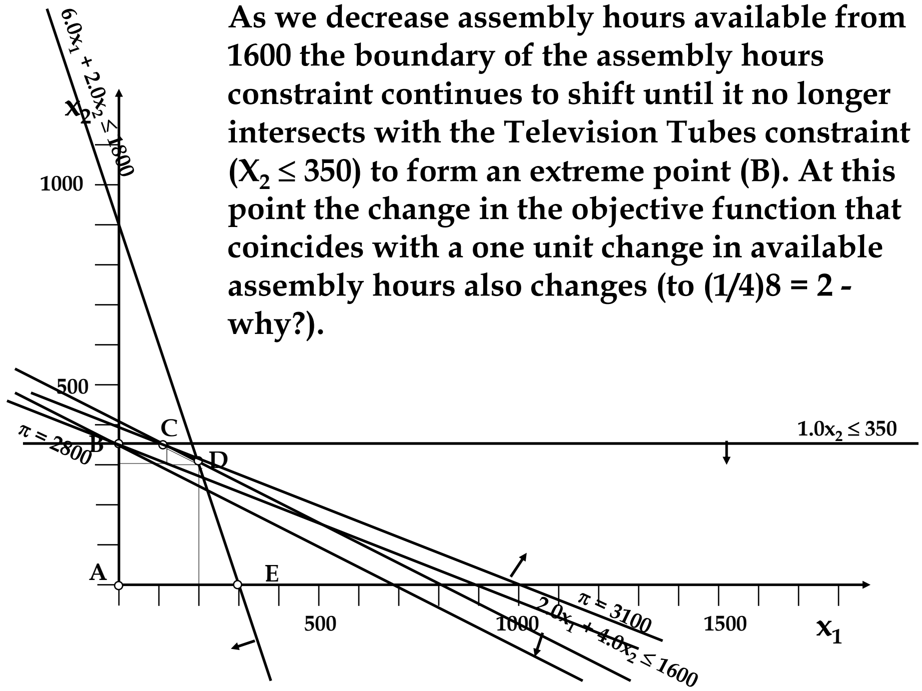
Produce 99.5 black & white sets, 350 color sets, and earn \$3098.5 profit

This again suggests that the shadow price for one unit of assembly labor is \$3098.5 - \$3100 = -\$1.50 (for a one unit *decrease* in assembly hours available) WITH RESPECT TO THE ORIGINAL FORMULATION!

Note that:

- **the shadow price is only effective over a limited range of values for available assembly labor hours in this direction as well (why?)**
- **shadow prices are only valid when considering the impact of changing a single constraint while holding all others constant**

As we decrease assembly hours available from 1600 the boundary of the assembly hours constraint continues to shift until it no longer intersects with the Television Tubes constraint ($X_2 \leq 350$) to form an extreme point (B). At this point the change in the objective function that coincides with a one unit change in available assembly hours also changes (to (1/4)8 = 2 - why?).



The shadow prices for each constraint in the Television Production Problem are:

Constraint	Shadow Price
assembly hours available	\$1.50
fabrication hours available	\$0.00
boundary for available color tubes	\$2.00
boundary for minimum color t.v. production	\$0.00

Finally note that:

- the shadow price corresponding to a constraint that is nonbinding at optimality must be \$0.00 (why)
- we will eventually develop methods for determining the range of values for a right-hand side over which the corresponding shadow price is valid.

3. **Primal Linear Program - the *original* standard formulation of a linear programming problem**
4. **Dual Linear Program - an *alternate* formulation of a linear programming problem that provides an interesting economic interpretation.**

Again consider the Television Production Problem as our Primal LP (ignoring the fourth constraint for simplification):

maximize $\pi = 3.0x_1 + 8.0x_2$

subject to:

$2.0x_1 + 4.0x_2 \leq 1600$ (# of assembly hours available)

$6.0x_1 + 2.0x_2 \leq 1800$ (# of fabrication hours available)

$1.0x_2 \leq 350$ (# of available color tubes)

$x_1, \quad x_2 \geq 0$ (nonnegativity)

Where x_1 **is the number of black & white sets produced**
x_2 **is the number of color sets produced**

Define separate variables u_j **to be the marginal value for one unit of what is represented by the** j^{th} **constraint in the primal formulation, i.e., for the Television Production Problem (again ignoring the fourth constraint for simplification):**

u_1 **is the marginal value of assembly time (per hour)**
u_2 **is the marginal value of processing time (per hour)**
u_3 **is the marginal value of color tubes (per tube)**

Then the marginal cost to make one Black & White set is

$$2u_1 + 6u_2$$

Similarly, the marginal cost to make one Color set is

$$4u_1 + 2u_2 + u_3$$

Why?

Economic theory dictates that:

- **at optimality we produce where marginal costs equal marginal revenues**
- **at suboptimal levels of production marginal costs exceed marginal revenues**

What are the marginal revenues produced by each product in this problem?

The marginal revenues produced by each product in this problem are the corresponding objective function coefficients from the c_i's or primal (original) formulation, i.e.,

Marginal Revenue for Black & White Sets = c_1 = 3.0
Marginal Revenue for Color Sets = c_2 = 8.0

Which results in the following relationships:

$$\underbrace{2u_1 + 6u_2}_{\text{Marginal Cost of a Black \& White Set}} \geq \underbrace{3.0}_{\text{Marginal Revenue of a Black \& White Set}}$$

and

$$\underbrace{4u_1 + 2u_2 + u_3}_{\text{Marginal Cost of a Color Set}} \geq \underbrace{8.0}_{\text{Marginal Revenue of a Color Set}}$$

Our overall cost to produce can be written

$$1600u_1 + 1800u_2 + 350u_3$$

so our objective when considering these new variables (U_j) that represent marginal costs per unit of production would be:

$$\text{Min } C = 1600u_1 + 1800u_2 + 350u_3$$

Thus we can rewrite our formulation as

maximize $C = 1600u_1 + 1800u_2 + 350u_3$

subject to: $2.0u_1 + 6.0u_2 \geq 3.0$ (cost of black & white sets)
$4.0u_1 + 2.0u_2 + 1.0u_3 \geq 8.0$ (cost of color sets)

Where u_1 is the number of black & white sets produced
u_2 is the number of color sets produced
u_3 is the number of color sets produced

Note that negative marginal costs would be illogical, so we should also incorporate nonegativity restrictions on the u_j's:

minimize $C = 1600u_1 + 1800u_2 + 350u_3$

subject to: $2.0u_1 + 6.0u_2 \geq 3.0$ (cost of black & white sets)
$4.0u_1 + 2.0u_2 + 1.0u_3 \geq 8.0$ (cost of color sets)
$u_1, \quad u_2, \quad u_3 \geq 0$ (nonnegativity)

Where u_1 is the number of black & white sets produced
u_2 is the number of color sets produced
u_3 is the number of color sets produced

This formulation, referred to as the Dual Linear Program, is mathematically equivalent to our primal formulation (and so yields the same optimal solution).

Note also that:

- **The number of constraints in the dual formulation is equal to the number of decision variables in the primal formulation**
- **The number of decision variables in the dual formulation is equal to the number of constraints in the primal formulation**
- **If the primal LP problem is a maximization (minimization) then the dual LP problem is a minimization (maximization)**
- **All information about the optimal primal solution can be recovered/derived from the optimal dual solution (and of course all information about the optimal dual solution can be recovered/derived from the optimal primal solution) and**
  - **primal decision variables are associated with dual constraints (and so dual slack & surplus variables)**
  - **primal constraints (and so slack & surplus variables) are associated with dual decision variables**

Observe the relationship between the primal and dual Note that negative marginal costs would be illogical, so we should also incorporate nonnegativity restrictions on the u_j's:

PRIMAL FORMULATION

maximize $\pi = 3.0x_1 + 8.0x_2$

subject to:

$$2.0x_1 + 4.0x_2 \leq 1600$$
$$6.0x_1 + 2.0x_2 \leq 1800$$
$$1.0x_2 \leq 350$$
$$x_1, \quad x_2 \geq 0$$

DUAL FORMULATION

minimize $C = 1600u_1 + 1800u_2 + 350u_3$

subject to:

$$2.0u_1 + 6.0u_2 \geq 3.0$$
$$4.0u_1 + 2.0u_2 + 1.0u_3 \geq 8.0$$
$$u_1, \quad u_2, \quad u_3 \geq 0$$

A similar relationship exists between the final tableaus for the primal and dual problems:

Unit Profit		3	8	0	0	0		
	Basic Mix	X_1	X_2	S_A	S_F	S_T	Solution (π)	Exchange Ratios
3	X_1	1.0	0.0	1/2	0.0	-2.0	100	
0	S_F	0.0	0.0	-3.0	1.0	10.0	500	
8	X_2	0.0	1.0	0.0	0.0	1.0	350	
	Sac.	3	8	3/2	0	2	3100	
	Imp.	0	0	-3/2	0	-2	-	

Unit Profit		1600	1800	350	0	0		
	Basic Mix	U_1	U_2	U_3	S_{BW}	S_C	Solution (C)	Exchange Ratios
350	U_3	0.0	-10.0	1.0	2.0	-1.0	2.0	
1600	U_1	1.0	3.0	0.0	-0.5	0.0	1.5	
	Sac.	1600	1300	350	-100	-350	3100	
	Imp.	0	500	0	100	350	-	

Generic Steps to Create the Dual Formulation:

- Create a dual variable u_j corresponding to each primal constraint number
- Use the primal right-hand side b_j as the dual objective function coefficient for for the i^{th} variable (u_j)
- Write the dual objective function - if the primal LP problem is a maximization (minimization) then the dual LP problem is a minimization (maximization)
- Use the primal constraint coefficient a_{ij} (j^{th} constraint and i^{th} variable) as the dual constraint coefficient a_{ji} (j^{th} variable and i^{th} constraint)

- **Use the primal objective function coefficient c_i as the right-hand side for the i^{th} dual constraint**
- **dual ≥ constraints correspond to primal slack variables;**
  dual ≤ constraints correspond to primal surplus variables;
  dual = constraints correspond to primal unrestricted variables;
  dual unrestricted variables correspond to primal = constraints.

Why be concerned with the dual formulation?

- **the dual may be easier than the primal to solve**
- **information from the optimal dual solution can be used to perform interesting economic analyses on the primal optimal solution**

For example, suppose we are considering production of a new product in the Television Production Problem. If production of one unit of the new product (a DVD player) requires 3 assembly hours and 5 fabrication hours, what is the minimum profit would we need to earn per unit to make the new product economically viable under current circumstances?

For example, suppose we are considering production of a new product in the Television Production Problem. If production of one unit of the new product (a DVD player) requires 3 assembly hours and 5 fabrication hours, what is the minimum profit would we need to earn per unit to make the new product economically viable under our current circumstances?

The opportunity cost of one DVD player would be

$$3u_1 + 5u_2 + 0u_3 = 3(1.5) + 5(0.0) + 0(3.0) = 4.5$$

so we should not manufacture the DVD player unless we can earn at least $4.50 profit per unit (i.e., marginal costs ≤ marginal profit)

5. Complimentary Slackness - the following *products* must all be zero at optimality:

   the *product* of a primal decision variable and the slack/ surplus variable from the associated dual constraint

   and

   the *product* of a dual decision variable and the slack/ surplus variable from the associated primal constraint

   For the Television Production Problem:

$$x_1 s_{BW} = 100*0 = 0.0$$
$$x_2 s_C = 350*0 = 0.0$$
$$s_A u_1 = 0*1.5 = 0.0$$
$$s_F u_2 = 500*0 = 0.0$$
$$s_T u_3 = 0*3.0 = 0.0$$

   Why must this so?

6. Sensitivity Analysis - assessment of the impact on the optimal solution of a change to the original formulation. Changes to the original formulation that are usually considered include:

- **changes in an *Objective Function Coefficient* (by how much can an objective function coefficient c_i change before the optimal values of the decision variables change?)**
- **changes in a *Right-Hand Side Value* b_j (how much must a right-hand side value change in order to force a change in the optimal set of basic variables?)**
- **the addition of a *new constraint* (how much will the addition of a new constraint affect the optimal solution?)**

- **changes in a *Constraint Coefficient* (by how much can a constraint coefficient a_{ij} change before the optimal values of the decision variables change?)**
- **making a basic variable take on a suboptimal value (by how will the objective function change if a basic variable's value is reduced to a suboptimal value?)**
- **the addition of a *new decision variable* (how much will the addition of a new decision variable affect the optimal solution?)**

We will focus on i) changes in an *Objective Function Coefficient*, ii) changes in a *Right-Hand Side Value*, and iii) the addition of a *new constraint*.

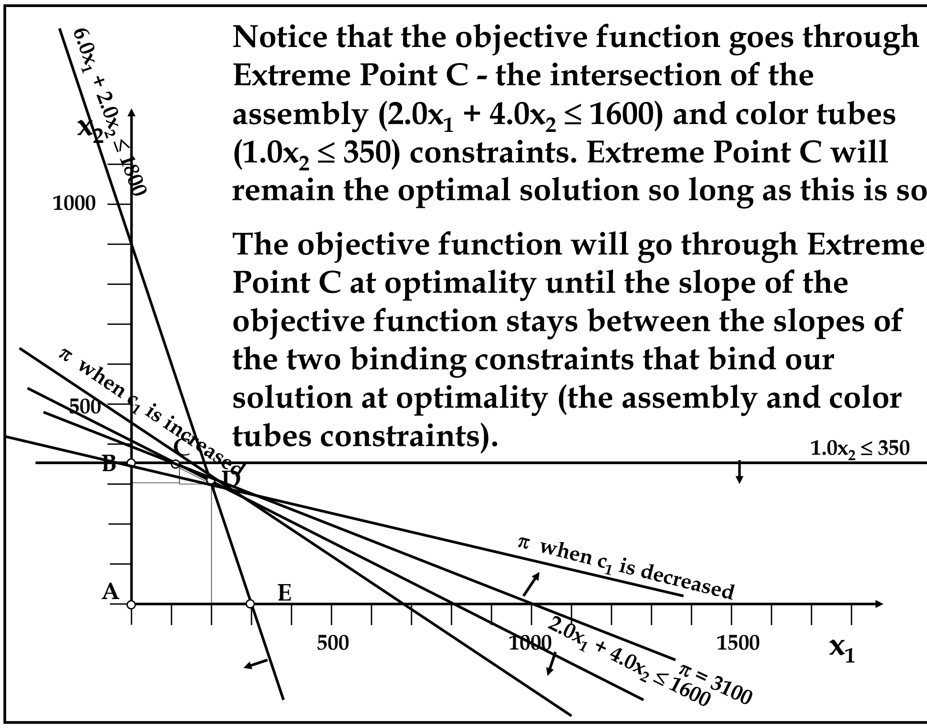
- Changes in an *Objective Function Coefficient* - when an objective function coefficient is altered, the slope of the objective function changes.

  Again consider the Television Production Problem as our Primal LP (ignoring the fourth constraint for simplification):

maximize $\pi = 3.0x_1 + 8.0x_2$

subject to: $2.0x_1 + 4.0x_2 \leq 1600$ (# of assembly hours available)
$6.0x_1 + 2.0x_2 \leq 1800$ (# of fabrication hours available)
$1.0x_2 \leq 350$ (# of available color tubes)
$x_1, \quad x_2 \geq 0$ (nonnegativity)

Where x_1 is the number of black & white sets produced
x_2 is the number of color sets produced



Notice that the objective function goes through Extreme Point C - the intersection of the assembly ($2.0x_1 + 4.0x_2 \leq 1600$) and color tubes ($1.0x_2 \leq 350$) constraints. Extreme Point C will remain the optimal solution so long as this is so.

The objective function will go through Extreme Point C at optimality until the slope of the objective function stays between the slopes of the two binding constraints that bind our solution at optimality (the assembly and color tubes constraints).

The slope of the objective function is $-c_1/c_2$.

The slopes of the assembly ($2.0x_1 + 4.0x_2 \leq 1600$) and color tubes ($1.0x_2 \leq 350$) constraints are:

assembly slope = $-a_{11}/a_{12} = -2.0/4.0 = -0.50$

color tubes slope = $-a_{31}/a_{32} = -0.0/1.0 = -0.00$

Thus we know that the objective function will go through Extreme Point E at optimality (and so the optimal values of x_1 and x_2 remain unchanged) as long as

$$-0.50 \leq -c_1/c_2 \leq 0.00$$

How much can we change the profit on black & white t.v. sets (c_1) while holding the profit on color t.v. sets (c_2) constant before the optimal values of the decision variables (x_1 & x_2) change?

If we hold the profit on color t.v. sets (c_2) constant at its current value of 8, we have

$$-0.50 \leq -c_1/8 \leq 0.00$$

which can be algebraically manipulated to yield

$$0.00 \leq c_1 \leq 4.00$$

Thus the profit on black & white t.v. sets (c_1) can range from 0.00 to 4.00 (while holding the profit on color t.v. sets c_2 constant) without changing the optimal values of the decision variables (x_1 & x_2).

Now how much can we change the profit on color t.v. sets (c_2) while holding the profit on black & white t.v. sets (c_1) constant before the optimal values of the decision variables (x_1 & x_2) change?

If we hold the profit on black & white t.v. sets (c_1) constant at its current value of 3, we have

$$-0.50 \leq -3/c_2 \leq 0.00$$

which can be algebraically manipulated to yield

$$6.00 \leq c_2 \leq \infty$$

Thus the profit on color t.v. sets (c_2) can range from 6.00 to ∞ (while holding the profit on black & white t.v. sets c_1 constant) without changing the optimal values of the decision variables (x_1 & x_2).

Alternate Approach to Calculating Allowable Changes in an *Objective Function Coefficient*

First calculate the *Improvement Ratio* - for a non-basic variable, its per unit improvement value divided by its Exchange Coefficient corresponding to the Basic Variable in question.

Now we can calculating Allowable Changes in an *Objective Function Coefficient*

For Decision Variables in the Basic Mix

Lower Sensitivity Limit = c_j - smallest absolute value of negative improvement ratios (or -∞ if no negative ratio exists)

Upper Sensitivity Limit = c_j + smallest positive improvement ratio (or ∞ if no positive ratio exists)

For Decision Variables not in the Basic Mix (for Maximizations)

Lower Sensitivity Limit = -∞

Upper Sensitivity Limit = c_j + absolute value of the corresponding per-unit improvement value

For Decision Variables not in the Basic Mix (for Minimizations)

Lower Sensitivity Limit = c_j - absolute value of the corresponding per-unit improvement value

Upper Sensitivity Limit = ∞

Example - Consider calculating the allowable changes in the *Objective Function Coefficient* for Black & White Sets (c_1) - which is in the basic mix.

Unit Profit		3	8	0	0	0		
	Basic Mix	X_1	X_2	S_A	S_F	S_T	Solution (π)	Exchange Ratios
3	X_1	1.0	0.0	1/2	0.0	-2.0	100	
0	S_F	0.0	0.0	-3.0	1.0	10.0	500	
8	X_2	0.0	1.0	0.0	0.0	1.0	350	
	Sac.	3	8	3/2	0	2	3100	
	Imp.	0	0	-3/2	0	-2	-	
	Imp. Ratio			$\frac{-3/2}{1/2}$	$\frac{0.0}{0.0}$	$\frac{-2}{-2.0}$		

For X_1 this gives us Improvement Ratio values of -3, 1, and ∞.

Now we can find the sensitivity limits for the *Objective Function Coefficient* associated with x_1 (c_1)

Lower Sensitivity Limit = c_1 - smallest absolute value of negative improvement ratios (or -∞ if no negative ratio exists) = 3 - |-3| = 0

Upper Sensitivity Limit = c_1 + smallest positive improvement ratio (or ∞ if no positive ratio exists) = 3 + 1 = 4

If we hold the profit on color t.v. sets (c_2) constant at its current value of 8, we have

$$0.00 \leq c_1 \leq 4.00$$

Thus the profit on black & white t.v. sets (c_1) can range from 0.00 to 4.00 (while holding the profit on color t.v. sets c_2 constant) without changing the optimal values of the decision variables (x_1 & x_2).

Example - Consider calculating the allowable changes in the *Objective Function Coefficient* for Color Sets (c_2) - which is in the basic mix.

Unit Profit		3	8	0	0	0		
	Basic Mix	X_1	X_2	S_A	S_F	S_T	Solution (π)	Exchange Ratios
3	X_1	1.0	0.0	1/2	0.0	-2.0	100	
0	S_F	0.0	0.0	-3.0	1.0	10.0	500	
8	X_2	0.0	1.0	0.0	0.0	1.0	350	
	Sac.	3	8	3/2	0	2	3100	
	Imp.	0	0	-3/2	0	-2	-	
	Imp. Ratio			$\frac{-3/2}{0.0}$	$\frac{0.0}{0.0}$	$\frac{-2}{1.0}$		

Now we can find the sensitivity limits for the *Objective Function Coefficient* associated with x_2 (c_2)

Lower Sensitivity Limit = c_2 - smallest absolute value of negative improvement ratios (or $-\infty$ if no negative ratio exists) = $8 - |-2| = 6$

Upper Sensitivity Limit = c_2 + smallest positive improvement ratio (or ∞ if no positive ratio exists) = ∞

If we hold the profit on color t.v. sets (c_2) constant at its current value of 8, we have

$$6.00 \leq c_2 \leq \infty$$

Thus the profit on color t.v. sets (c_2) can range from 0.00 to ∞ (while holding the profit on black & white t.v. sets c_1 constant) without changing the optimal values of the decision variables (x_1 & x_2).

- **Changes in a *Right-Hand Side Value* - as we have already seen, this is related to the concept of *Shadow Price*.**

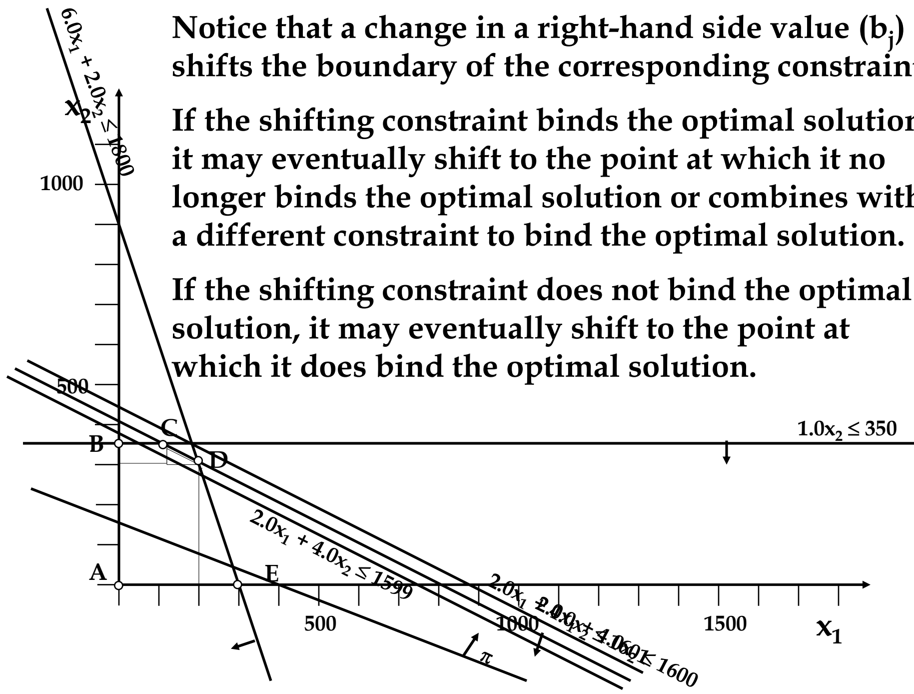
  Again consider the Television Production Problem as our Primal LP (ignoring the fourth constraint for simplification):

maximize $\pi = 3.0x_1 + 8.0x_2$

subject to:

$$2.0x_1 + 4.0x_2 \leq 1600 \text{ (\# of assembly hours available)}$$
$$6.0x_1 + 2.0x_2 \leq 1800 \text{ (\# of fabrication hours available)}$$
$$1.0x_2 \leq 350 \text{ (\# of available color tubes)}$$
$$x_1, \quad x_2 \geq 0 \text{ (nonnegativity)}$$

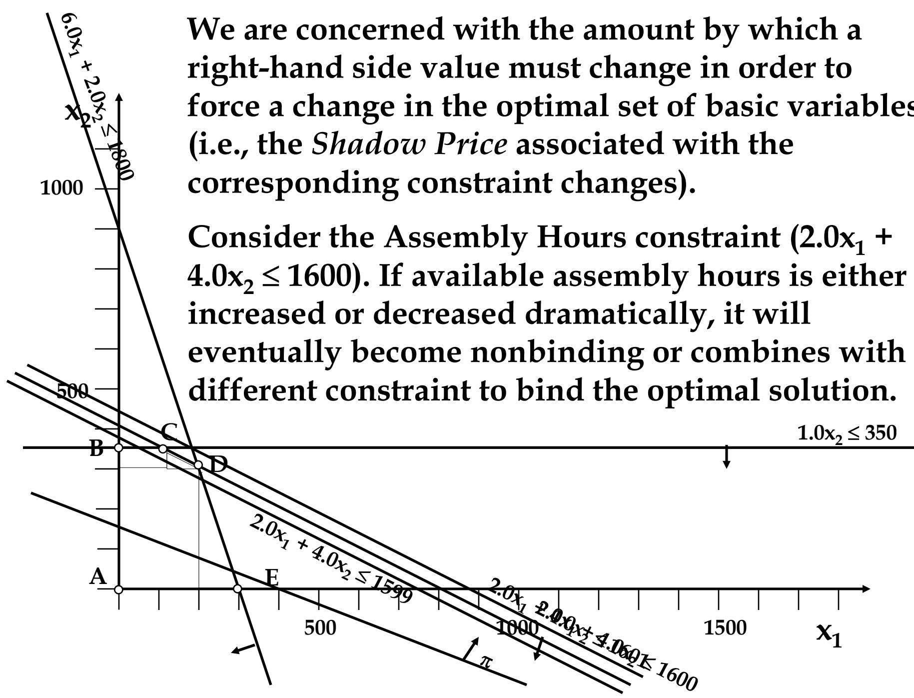
Where x_1 **is the number of black & white sets produced**
x_2 **is the number of color sets produced**



Notice that a change in a right-hand side value (b_j) shifts the boundary of the corresponding constraint.

If the shifting constraint binds the optimal solution, it may eventually shift to the point at which it no longer binds the optimal solution or combines with a different constraint to bind the optimal solution.

If the shifting constraint does not bind the optimal solution, it may eventually shift to the point at which it does bind the optimal solution.



We are concerned with the amount by which a right-hand side value must change in order to force a change in the optimal set of basic variables (i.e., the *Shadow Price* associated with the corresponding constraint changes).

Consider the Assembly Hours constraint ($2.0x_1 + 4.0x_2 \le 1600$). If available assembly hours is either increased or decreased dramatically, it will eventually become nonbinding or combines with a different constraint to bind the optimal solution.

Assembly Hours constraint ($2.0x_1 + 4.0x_2 \le 1600$) will become nonbinding with respect to the optimal solution when available assembly hours is increased past the point that the intersection of the fabrication ($6.0x_1 + 2.0x_2 \le 1800$) and color tubes ($1.0x_2 \le 350$) constraint boundaries bind the optimal solution.

Thus we must solve for the intersection of the fabrication and color tubes constraint boundaries and substitute the resulting values of decision variables (x_1 and x_2) into the left-hand side of the Assembly Hours constraint ($2.0x_1 + 4.0x_2$) to determine the maximum value the Assembly Hours constraint's) right hand side can be before the fabrication ($6.0x_1 + 2.0x_2 \le 1800$) and color tubes ($1.0x_2 \le 350$) constraints bind the optimal solution.

At the intersection of the fabrication and color tubes constraint boundaries we have that

$$x_1 = 183.33 \text{ and } x_2 = 350$$

Substitution of these values into the left-hand side of the Assembly Hours constraint yields

$$2.0x_1 + 4.0x_2 = 2.0(183.33) + 4.0(350) = 1766.67$$

Thus the Assembly Hours constraint will become nonbinding with respect to the optimal solution when available assembly hours is increased beyond 1766.67 (we could not use more than 1766.67 assembly hours under current conditions, and so the shadow price of assembly hours would become zero at this point).

Assembly Hours constraint ($2.0x_1 + 4.0x_2 \leq 1600$) will combine with the x_2 axis (instead of the color tubes constraint) to bind the optimal solution when available assembly hours is reduced past the point that the intersection of the x_2 axis ($x_1 \geq 0$) and color tubes ($1.0x_2 \leq 350$) constraint boundaries bind the optimal solution.

Thus we must solve for the intersection of the x_2 axis and color tubes constraint boundaries and substitute the resulting values of decision variables (x_1 and x_2) into the left-hand side of the Assembly Hours constraint ($2.0x_1 + 4.0x_2$) to determine the minimum value the Assembly Hours constraint's) right hand side can be before the x_2 axis ($x_1 \geq 0$) and color tubes ($1.0x_2 \leq 350$) constraint bind the optimal solution.

At the intersection of the x_2 axis and color tubes constraint boundaries we have that

$$x_1 = 0 \text{ and } x_2 = 350$$

Substitution of these values into the left-hand side of the Assembly Hours constraint yields

$$2.0x_1 + 4.0x_2 = 2.0(0) + 4.0(350) = 1400$$

Thus the Assembly Hours constraint will combine with the x_2 axis (instead of the color tubes constraint) to bind the optimal solution with respect to the optimal solution when available assembly hours is reduced below 1400 (if we have less than 1400 assembly hours under current conditions, we would not be able to use all 350 color tubes so the shadow price of assembly hours would change at this point).

Now we know the current shadow price of assembly hours ($1.50) is valid while the right-hand side of the Assembly Hours constraint (i.e., assembly hours available) satisfies

$$1400 \geq b_1 \geq 1766.67$$

Some software report the allowable change in the right hand side of a constraint, i.e.,

$$-200 \geq \Delta b_1 \geq 166.67$$

Of course, both tell you the same thing!

This approach works well, but is obviously dependent on the graphical approach - what can we do if there are more than two decision variables?

- **Addition of a *New Constraint* - this is a matter of considering whether the new constraint would be met at the current optimal solution.**

  Again consider the Television Production Problem as our Primal LP (ignoring the fourth constraint for simplification):

maximize $\pi = 3.0x_1 + 8.0x_2$

subject to: $2.0x_1 + 4.0x_2 \leq 1600$ **(# of assembly hours available)**
$6.0x_1 + 2.0x_2 \leq 1800$ **(# of fabrication hours available)**
$1.0x_2 \leq 350$ **(# of available color tubes)**
$x_1, \quad x_2 \geq 0$ **(nonnegativity)**

Where x_1 **is the number of black & white sets produced**
x_2 **is the number of color sets produced**

Suppose we are informed that we are running short of 19 inch television consoles and have only 400 available. Since both products are 19 inch television sets, the new consoles constraint would look like this:

$1.0x_1 + 1.0x_2 \leq 400$ (# of consoles available)

Must we alter our current optimal solution to satisfy this new constraint?

Our current optimal solution is $X_1 = 100$, $X_2 = 350$, $S_A = 0$, $S_F = 500$, $S_T = 0$, and $\pi = 3100$. For the consoles constrain this would imply

$$1.0(100) + 1.0(350) > 400$$

The new constraint would not be satisfied by the current optimal solution - we must resolve the problem with the new constraint.

What would happen if we had 600 consoles available?