

II. Linear Programming

A Quick Example

Suppose we own and manage a small manufacturing facility that produced television sets.

- What would be our organization's immediate goal?
- On what would our relative achievement of that goal depend?
- What would limit/obstruct our relative achievement of that goal?

A. Basic Concepts & Definitions

1. **Objective Function** - mathematical relationship between parameters and decision variables that represents relative effectiveness of proposed solutions to a problem (this is what we usually wish to maximize or minimize).
2. **Constraint** - mathematical relationship between parameters and decision variables that represents a limitation on objective function.
3. **Objective Function Coefficient** - parameter that represents the change in the objective function that results in a one-unit change in the value of the corresponding decision variable. Note that the objective function coefficient for the j^{th} decision variable is usually denoted c_j .

4. **Constraint Coefficient** - parameter that represents the change in the achieved value (right hand side) of a constraint that results in a one-unit change in the value of the corresponding decision variable. Note that the constraint coefficient the j^{th} decision variable in the i^{th} constraint is sometimes denoted e_{ij} (or often denoted a_{ij}).
5. **Right-Hand Side Value** - parameter that represents the limitation on the achieved value (right hand side) of the corresponding constraint. Note that the right-hand side value constraint for the i^{th} constraint is usually denoted b_i .
6. **Formulation** - mathematical expression of a given situation or system

Now let's review the TV problem and identify (i) the parameters and (ii) the decision variables:

A production firm makes 19' black & white and color television sets. It takes 6 hours to fabricate the components and 2 hours to assemble a black & white sets. On the other hand, color sets require 2 hours to fabricate the components and 4 hours to assemble. Only 1800 fabrication hours and 1600 assembly hours are available during any time period. Additionally, the firm has only 350 color picture tubes available, and they have a prior order for 75 color televisions that they need to fill. Color televisions can be produced for \$242 and sold for \$250, while black & white sets can be produced for \$147 and sold for \$150. Define the DV's, develop an objective function, and form the appropriate constraints.

The *Decision Variables* are:

x_1 is the number of black & white sets produced

x_2 is the number of color sets produced

Other relevant information includes:

Parameters	x_1	x_2	RHS
Objective Function Coefficients	3.0	8.0	---
# of Assembly Hours Constraint Coefficients	2.0	4.0	1600
# of Fabrication Hours Constraint Coefficients	6.0	2.0	1800
# of Fabrication Hours Constraint Coefficients	0.0	1.0	350
Minimum # of Color Sets Constraint Coefficients	0.0	1.0	75

7. **Mathematical Programming** - mathematical approach to finding the optimal value of an objective function while satisfying constraints on the values of the decision variables.

8. **Linear Programming** - mathematical programming where the objective function and constraints are linear (i.e., the change in any of these functions that coincides with a one unit change in any decision variable is the same no matter what value the decision variables has taken). A *Mathematical Programming* problem must meet the conditions of *Strict Linearity* to be a *Linear Programming* problem:

9. Conditions of Strict Linearity

- **Deterministic Nature** - Exact knowledge of parameters
- **Additivity** - variables must be measured in comparable units within the objective function and the constraints
- **Direct Proportionality** - Relationships expressed by objective function and constraints must be linear and constant
- **Fractionality** - DV values must be permitted to assume any fractional numerical value

In addition, LP problems must satisfy

- **Nonnegativity**
- **Single goal**

10. Integer Programming - mathematical programming where the values of some or all decision variables are limited to whole numbers.

11. Goal Programming - mathematical programming with multiple objectives.

12. Nonlinear Programming - mathematical programming where the objective function and/or constraints are nonlinear.

13. Stochastic Programming - mathematical programming where the values of some or all parameters are not known with certainty, but the probability distributions of these unknown parameters are known.

14. Statistical Programming - mathematical programming where the values of some or all parameters are not known with certainty, and their values are estimated using sample data.

15. Dynamic Programming - mathematical programming where the values of some decision variables are dependent on the values of other decision variables. This interdependency among decision variables usually take the form of sequential, or multi-stage, decision problems; hence, the name "dynamic" programming.

B. Formulating LP Problems

- 1. Suggested Steps in Formulating ANY problem (LP or other)**
 - Gain a thorough understanding of the problem scenario
 - Determine the general goal (maximize, minimize; costs, profits, etc.) of the problem
 - Define and label the Decision Variables and determine how they impact the goal of the problem
 - Write the goal of the problem in functional form utilizing your decision variables (i.e., write your Objective Function)
 - Identify and label the factors that restrict achievement of your Objective (i.e., write your constraints)

Now let's review the TV problem and our previously identified (i) parameters and (ii) decision variables to compose a formulation:

A production firm makes 19' black & white and color television sets. It takes 6 hours to fabricate the components and 2 hours to assemble a black & white sets. On the other hand, color sets require 2 hours to fabricate the components and 4 hours to assemble. Only 1800 fabrication hours and 1600 assembly hours are available during any time period. Additionally, the firm has only 350 color picture tubes available, and they have a prior order for 75 color televisions that they need to fill. Color televisions can be produced for \$242 and sold for \$250, while black & white sets can be produced for \$147 and sold for \$150. Define the DV's and identify the parameters for this problem.

$$\text{maximize } \pi = 3.0x_1 + 8.0x_2$$

$$\text{subject to: } 2.0x_1 + 4.0x_2 \leq 1600 \text{ (\# of assembly hours available)}$$

$$6.0x_1 + 2.0x_2 \leq 1800 \text{ (\# of fabrication hours available)}$$

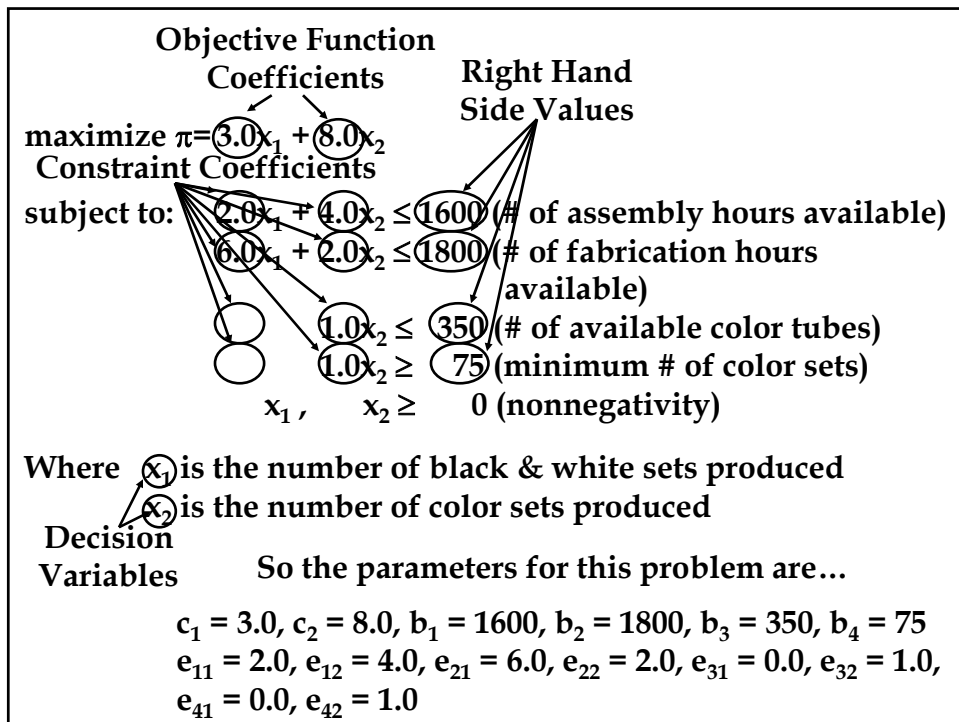
$$1.0x_2 \leq 350 \text{ (\# of available color tubes)}$$

$$1.0x_2 \geq 75 \text{ (minimum \# of color sets)}$$

$$x_1, x_2 \geq 0 \text{ (nonnegativity)}$$

Where x_1 is the number of black & white sets produced
 x_2 is the number of color sets produced

THIS CONSTITUTES A COMPLETE AND VALID FORMULATION!



C. Solving LP Problems

1. Trial & Error - change values of decision variables until we find an *acceptable* solution.

Note that we can accomplish this by putting our formulation into Excel

- this approach *will not guarantee* optimality
- however, this approach *is superior to no systematic evaluation* of the problem

2. Intelligent Enumeration - use the Extreme Point Theorem to limit our search of feasible solutions.

The Extreme Point Theorem (The Fundamental Rule of Linear Programming) - If an optimal solution to a linear programming problem exists, it must occur at one of the extreme points (a feasible intersections of constraint boundary lines).

Geometrically this implies that we only need to consider a finite number of potential solutions to any problem.

Let's find the values of the decision variables at the intersection of the boundary for assembly hours available and the boundary for fabrication hours available is point by simultaneously solving for the constraint boundary lines that intersect at that point

$$\begin{array}{r} 2.0x_1 + 4.0x_2 = 1600 \\ -2(6.0x_1 + 2.0x_2 = 1800) \\ \hline -10.0x_1 + 0.0x_2 = -2000 \rightarrow x_1 = 200 \end{array}$$

which implies that

$$2.0x_1 + 4.0x_2 = 2.0(200) + 4.0x_2 = 1600 \rightarrow 4.0x_2 = 1200 \rightarrow x_2 = 300$$

So we have $x_1 = 200$ and $x_2 = 300$ at this point.

If we substitute these values of the decision variables into the objective function we can determine the value of the solution at this point

$$\pi = 3.0x_1 + 8.0x_2 = 3.0(200) + 8.0(300) = 3000$$

Finally, we must check to determine if this solution is feasible (i.e., satisfies all constraints):

At this point

$$x_1 \text{ (the number of black \& white sets produced)} = 200$$

$$x_2 \text{ (the number of color sets produced)} = 300$$

so

$$2.0x_1 + 4.0x_2 = 2.0(200) + 4.0(300) = 1600 \leq 1600 \text{ (\# of assembly hours available)}$$

$$6.0x_1 + 2.0x_2 = 6.0(200) + 2.0(300) = 1800 \leq 1800 \text{ (\# of fabrication hours available)}$$

$$1.0x_2 = 1.0(300) = 300 \leq 350 \text{ (\# of available color tubes)}$$

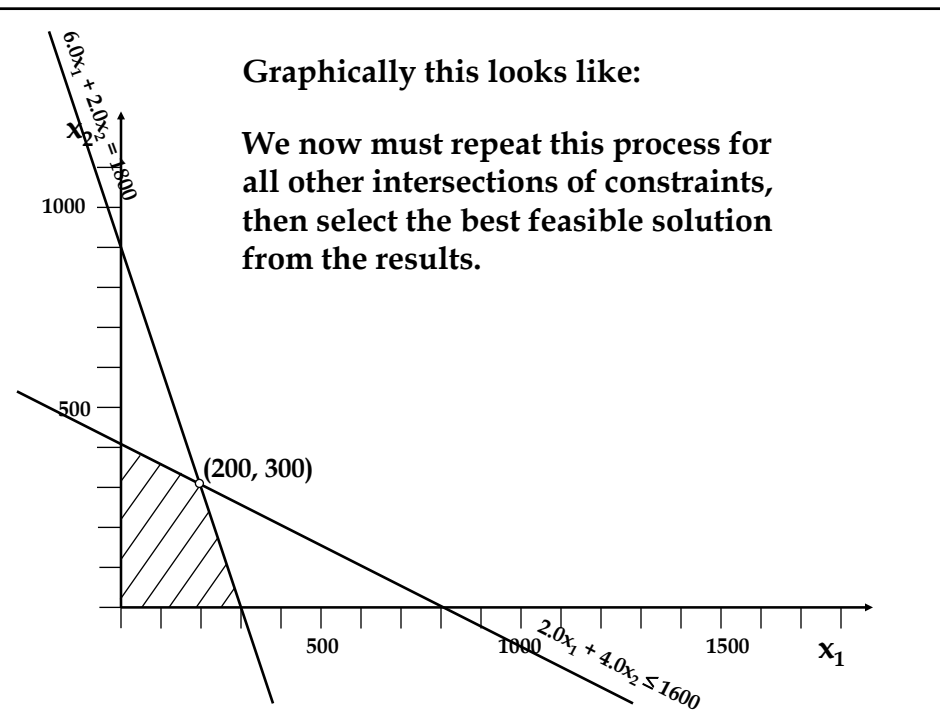
$$1.0x_2 = 1.0(300) = 300 \geq 75 \text{ (minimum \# of color sets)}$$

$$x_1, x_2 = 200, 300 \geq 0 \text{ (nonnegativity)}$$

All constraints are met - $x_1 = 200, x_2 = 300$ is a feasible solution.

Graphically this looks like:

We now must repeat this process for all other intersections of constraints, then select the best feasible solution from the results.



Constraints Intersected		DV's		O.F.	Constraints Satisfied?			
		X_1	X_2	π	Assembly	Fabrication	Color Tubes	Color TV's
Assembly	Fabrication	200	300	3000	Yes	Yes	Yes	Yes
Assembly	Color Tubes	100	350	3100	Yes	Yes	Yes	Yes
Assembly	Color TV's	650	75	2550	Yes	No	Yes	Yes
Assembly	X_1 axis	0	400	3200	Yes	Yes	No	Yes
Assembly	X_2 axis	800	0	2400	Yes	No	Yes	No
Fabrication	Color Tubes	183	350	3350	No	Yes	Yes	Yes
Fabrication	Color TV's	275	75	1425	Yes	Yes	Yes	Yes
Fabrication	X_1 axis	0	900	5400	No	Yes	No	Yes
Fabrication	X_2 axis	300	0	900	Yes	Yes	Yes	No
Color Tubes	X_1 axis	0	350	2800	Yes	Yes	Yes	Yes
Color TV's	X_1 axis	0	75	600	Yes	Yes	Yes	Yes

As you can see, this can get very complicated if you have many decision variables and constraints

- **some constraints are parallel (they do not intersect)**
- **occasionally multiple constraints intersect at the same point**
- **the number of intersections for which we must solve grows very rapidly as the number of constraints increases**
- **the difficulty of solving the intersections simultaneously grows as the number of decision variables increases**
- **you have to do a lot of work on solutions that are ultimately infeasible**

3. Graphical Approach - use a plot of the constraints and objective function to find the *optimal* solution.

- this approach *will guarantee* optimality
- you will only work on a subset of the feasible extreme points
- however, this approach will only work for problems with two decision variables
- this approach does lend insight into the geometry of other solution algorithms

Steps in the Graphical Approach to Solving LP Problems

- formulate the LP problem
- plot the boundary lines for each constraint (i.e., the constraint lines) by
 - i) rewriting a constraint as an equality
 - ii) finding two points that satisfy the rewritten constraint, and
 - iii) drawing a straight line through the two points
- find the area (valid side of the boundary lines) for each constraint by selecting a convenient point on either side of the boundary and checking to see if it satisfies the original constraint.
- identify the feasible region (intersection of all areas identified in the previous step) and Extreme Points (all feasible intersections of constraint boundary lines).

- plot the objective function for two possible values and determine the direction of improvement
 - i) set the objective function equal to some mathematically convenient value (that hopefully places the objective function line through the feasible region)
 - ii) find two points that satisfy the rewritten objective function, and
 - iii) draw a straight line through the two points
 - iv) repeat steps i-iii for some other mathematically convenient value, and
 - v) find which the direction of improvement (orthogonally away from objective function with the lower value/toward the objective function with the greater value)

- move in a parallel fashion in the direction of improvement until you hit the last point in the feasible region - this is the optimal solution.
- Find the values of the decision variables at this point by simultaneously solving for the constraint boundary lines that intersect at the optimal point
- substitute the optimal values of the decision variables (found in the previous step) into the objective function to determine the optimal solution

Now let's solve the TV problem and graphically:

- first formulate the LP problem (we have already done so)

$$\text{maximize } \pi = 3.0x_1 + 8.0x_2$$

$$\begin{aligned} \text{subject to: } & 2.0x_1 + 4.0x_2 \leq 1600 \text{ (# of assembly hours available)} \\ & 6.0x_1 + 2.0x_2 \leq 1800 \text{ (# of fabrication hours available)} \\ & 1.0x_2 \leq 350 \text{ (# of available color tubes)} \\ & 1.0x_2 \geq 75 \text{ (minimum # of color sets)} \\ & x_1, x_2 \geq 0 \text{ (nonnegativity)} \end{aligned}$$

Where x_1 is the number of black & white sets produced
 x_2 is the number of color sets produced

- plot the boundary lines for each constraint (i.e., the constraint lines) by

i) rewriting a constraint as an equality

$$\begin{aligned} 2.0x_1 + 4.0x_2 &= 1600 \text{ (boundary for assembly hours available)} \\ 6.0x_1 + 2.0x_2 &= 1800 \text{ (boundary for fabrication hours available)} \\ 1.0x_2 &= 350 \text{ (boundary for available color tubes)} \\ x_1, x_2 &= 0 \text{ (boundary for nonnegativity)} \end{aligned}$$

ii) finding two points that satisfy the rewritten constraint

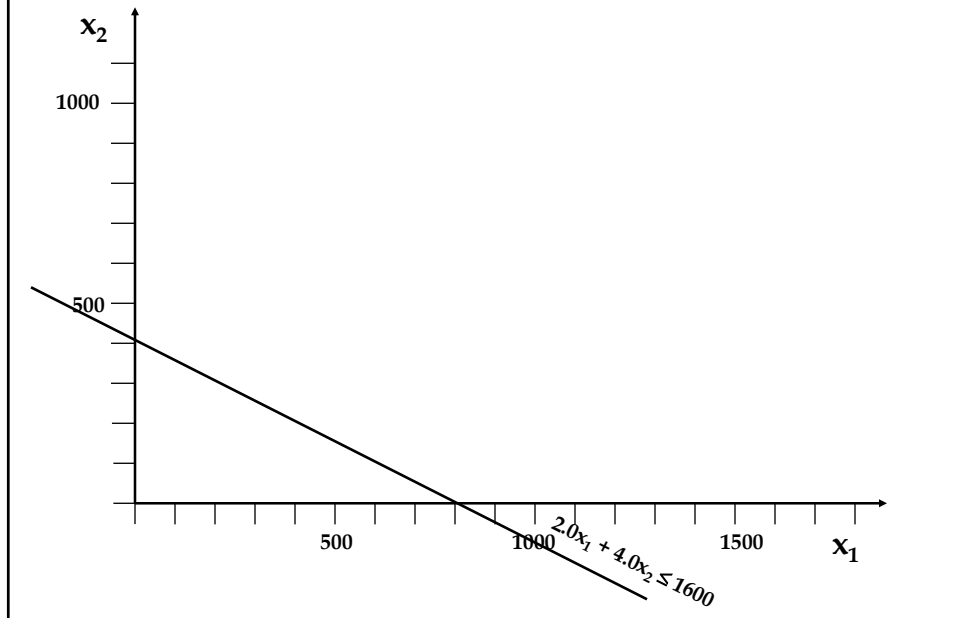
Let's work with the assembly hours first:

$$2.0x_1 + 4.0x_2 = 1600$$

x_1	x_2
0	400
800	0

So the points (0, 400) and (800, 0) lie on the boundary of this constraint

iii) draw a straight line through the two points

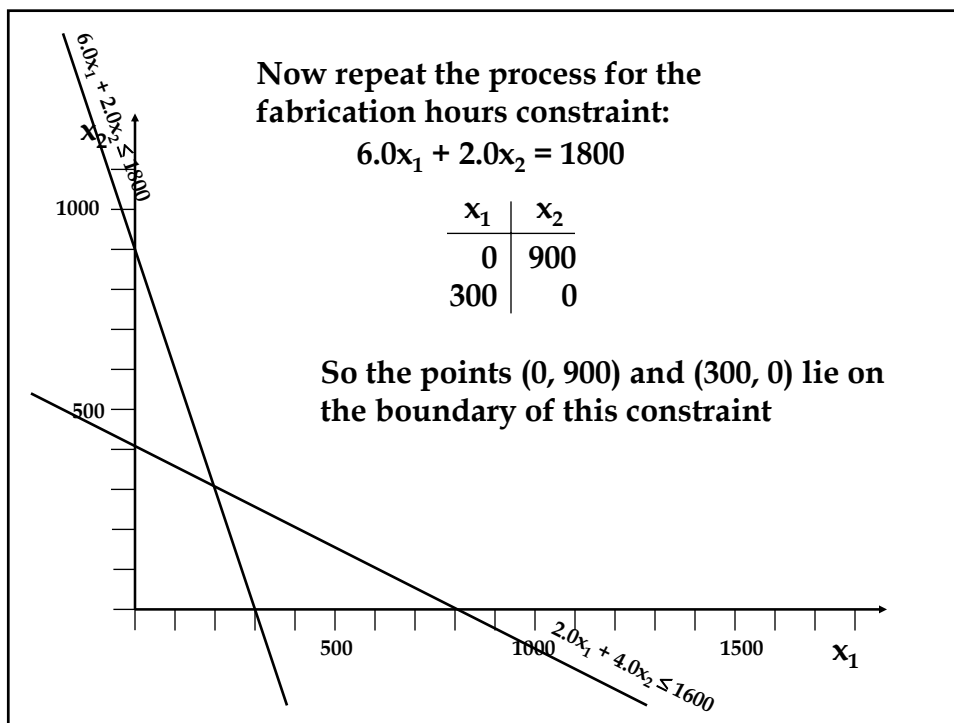


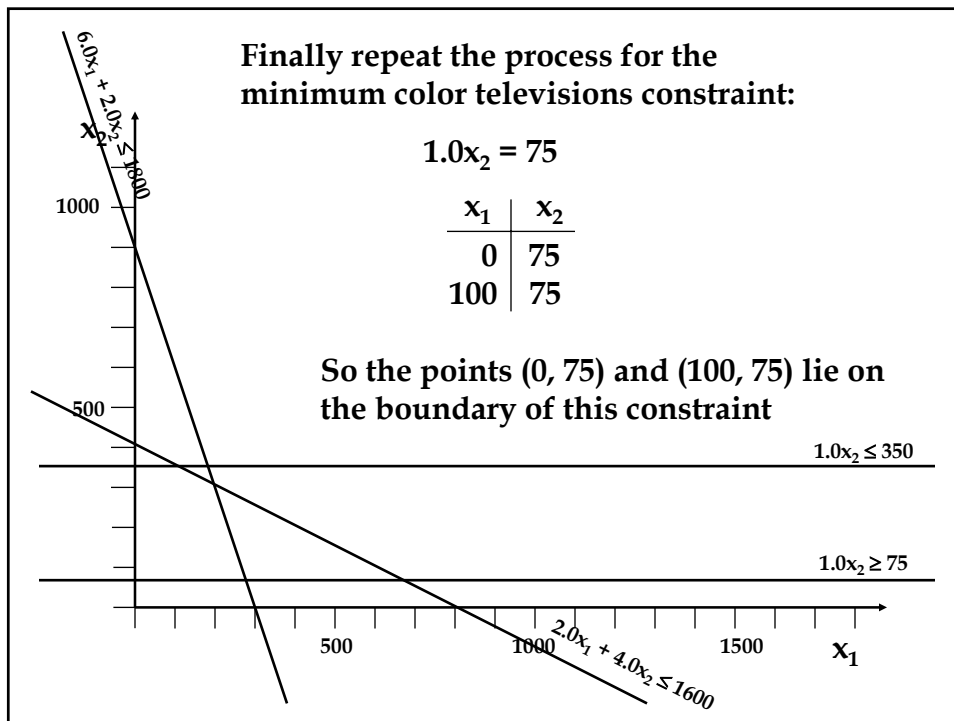
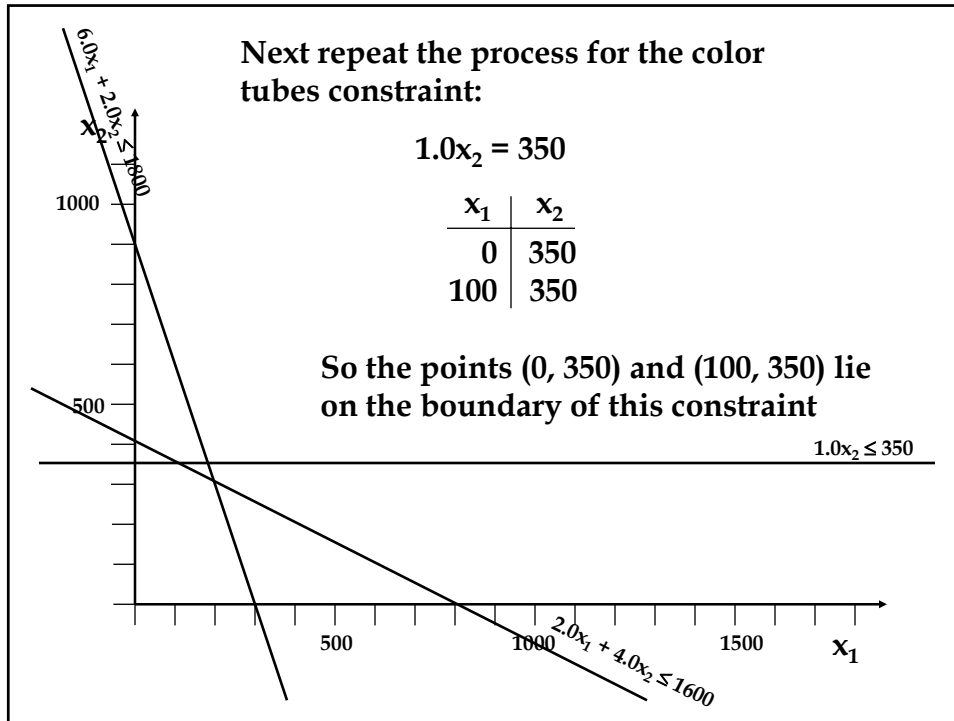
Now repeat the process for the fabrication hours constraint:

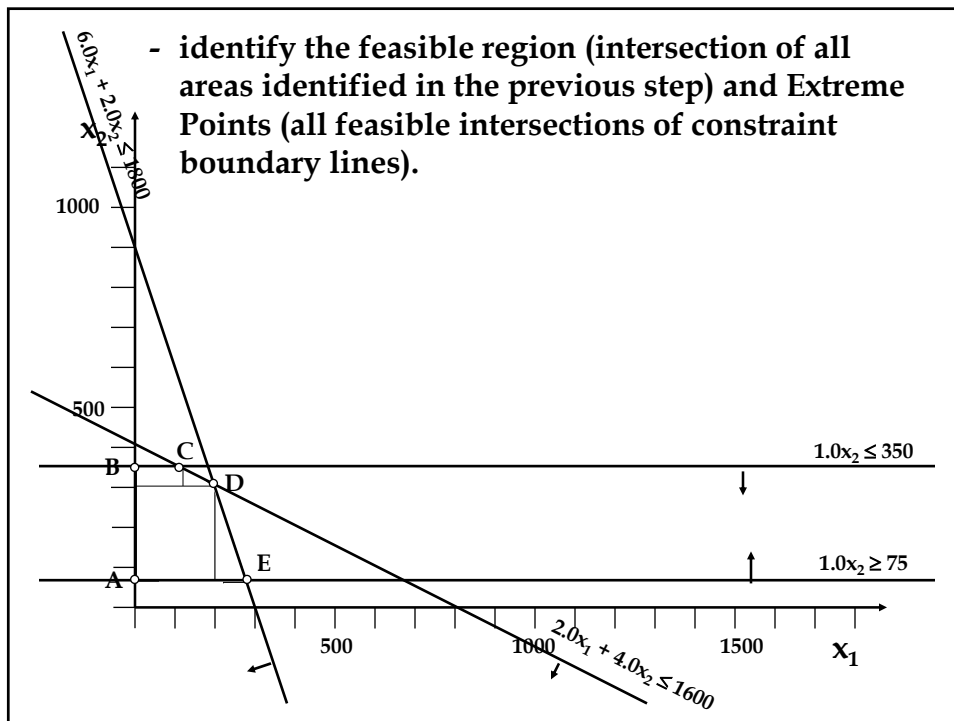
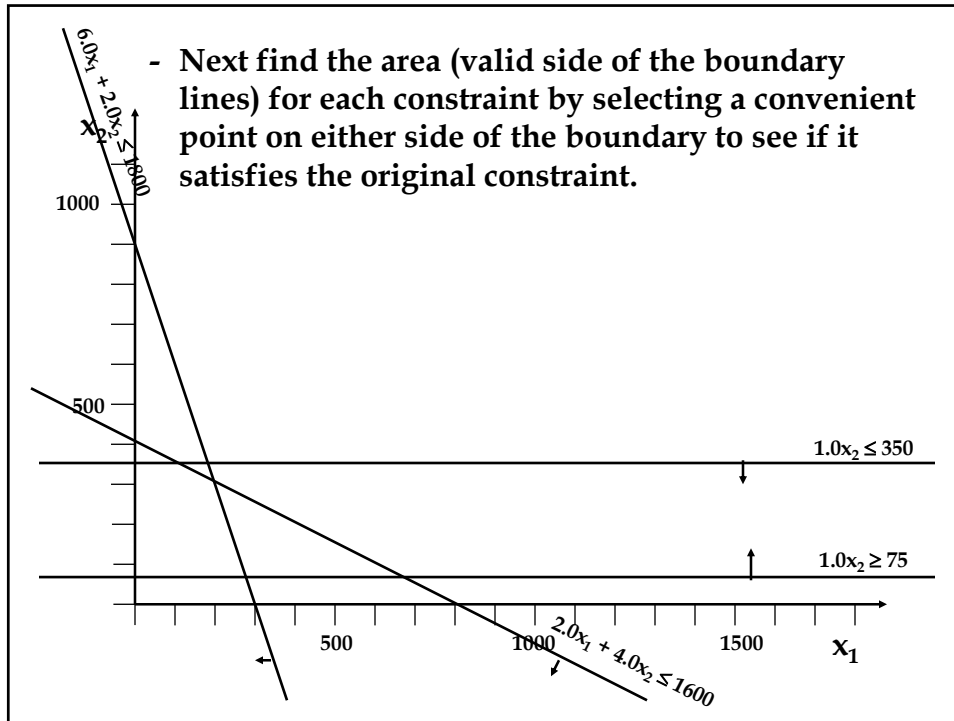
$$6.0x_1 + 2.0x_2 = 1800$$

x_1	x_2
0	900
300	0

So the points (0, 900) and (300, 0) lie on the boundary of this constraint







- plot the objective function for two possible values and determine the direction of improvement

- i) set the objective function equal to some mathematically convenient value (that hopefully places the objective function line through the feasible region) and
- ii) find two points that satisfy the rewritten objective function

$$1200 = 3.0x_1 + 8.0x_2$$

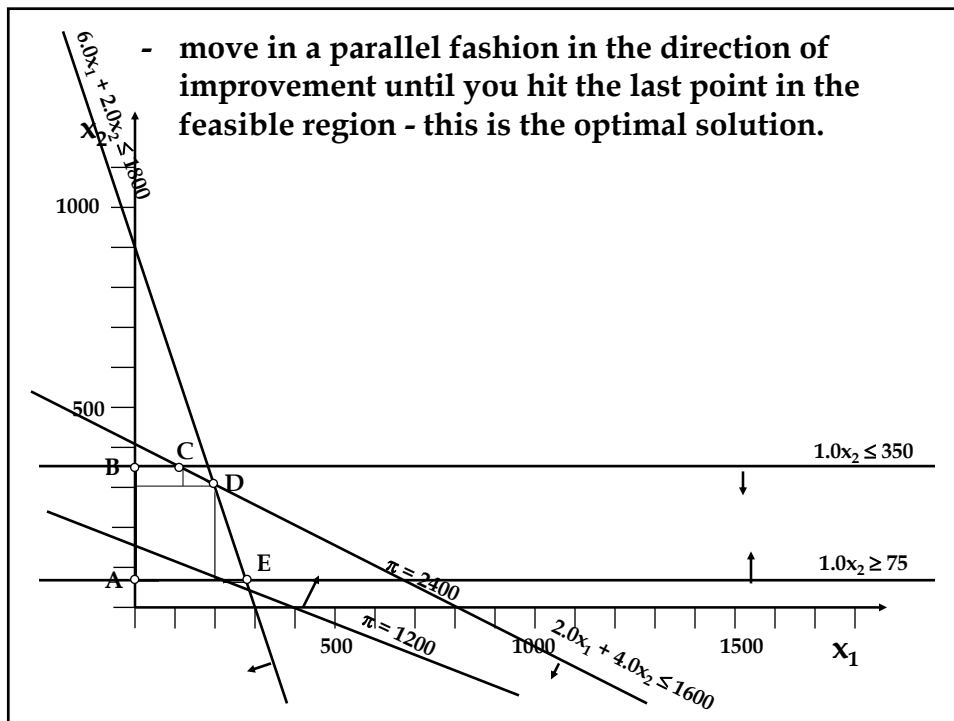
x_1	x_2
0	150
400	0

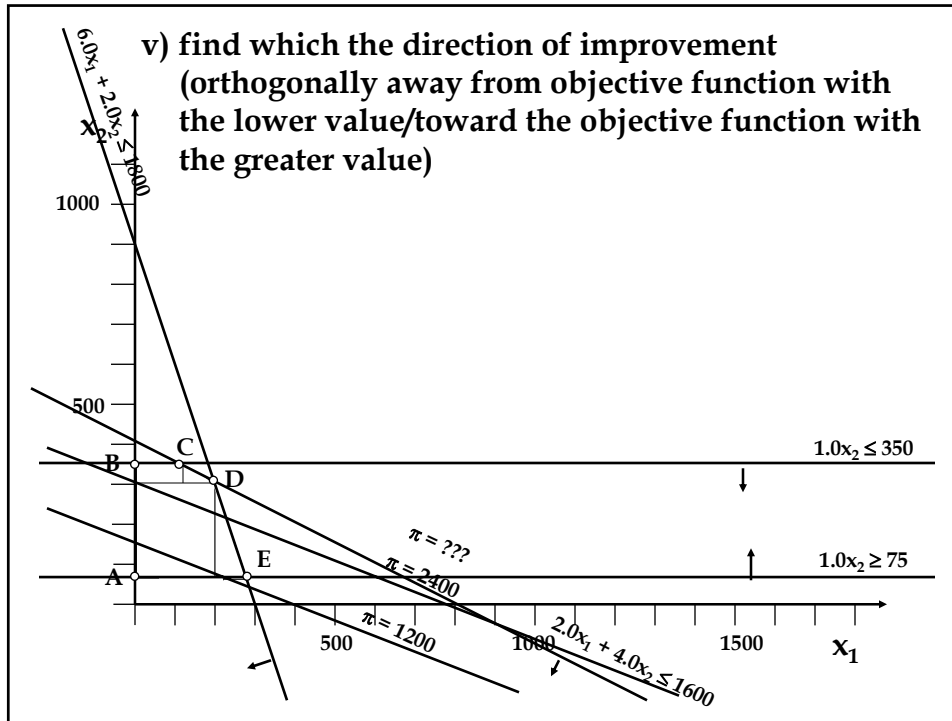
$$2400 = 3.0x_1 + 8.0x_2$$

x_1	x_2
0	300
800	0

So the points (0, 150) and (400, 0) lie on this constraint line

So the points (0, 300) and (800, 0) lie on this constraint line





- Find the values of the decision variables at this point by simultaneously solving for the constraint boundary lines that intersect at the optimal point

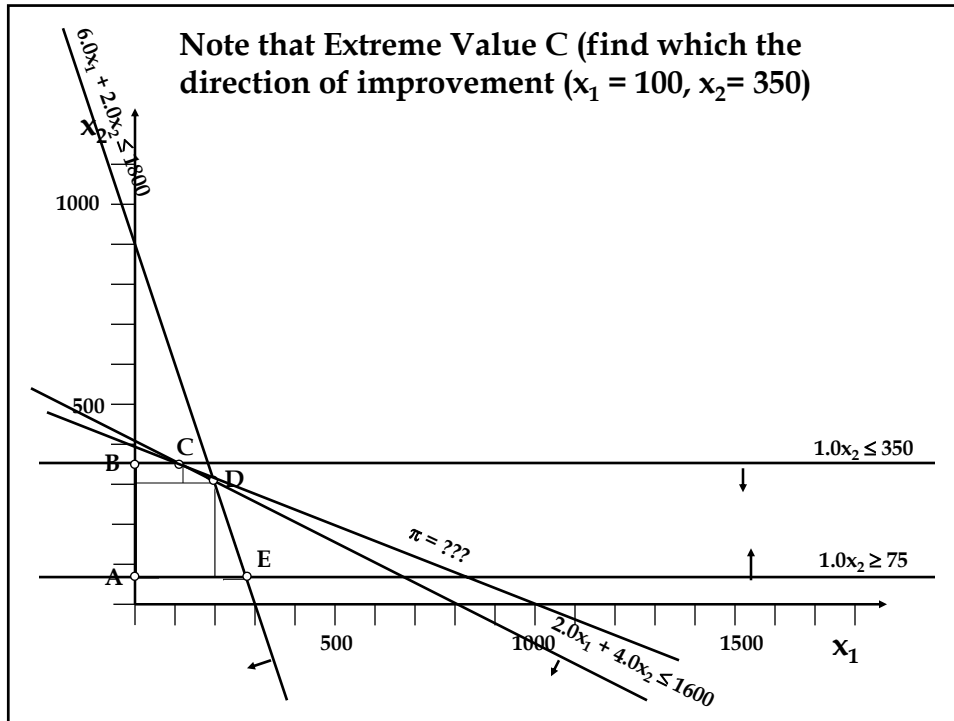
$$\begin{array}{r}
 2.0x_1 + 4.0x_2 = 1600 \\
 -4(\quad 1.0x_2 = 350) \\
 \hline
 2.0x_1 + 0.0x_2 = 200 \rightarrow x_1 = 100
 \end{array}$$

which implies that

$$2.0x_1 + 4.0x_2 = 2.0(100) + 4.0x_2 = 1600 \rightarrow 4.0x_2 = 1400 \rightarrow x_2 = 350$$

Note that this problem could have been solved even more easily if we observed that $x_2 = 350$ at Extreme Point C!

- substitute the optimal values of the decision variables (found in the previous step) into the objective function to determine the optimal solution



- substitute the optimal values of the decision variables (found in the previous step) into the objective function to determine the optimal solution

$$\pi = 3.0x_1 + 8.0x_2 = 3.0(100) + 8.0(350) = 3100$$

So the optimal solution is:

Produce 100 black & white sets, 350 color sets, and earn \$3100 profit

What if the last point in the feasible region that you hit isn't a point where constraint boundary lines intersect?

Geometrically the last point in the feasible region that you hit *must be* a point where constraint boundary lines intersect.

How do we know that?

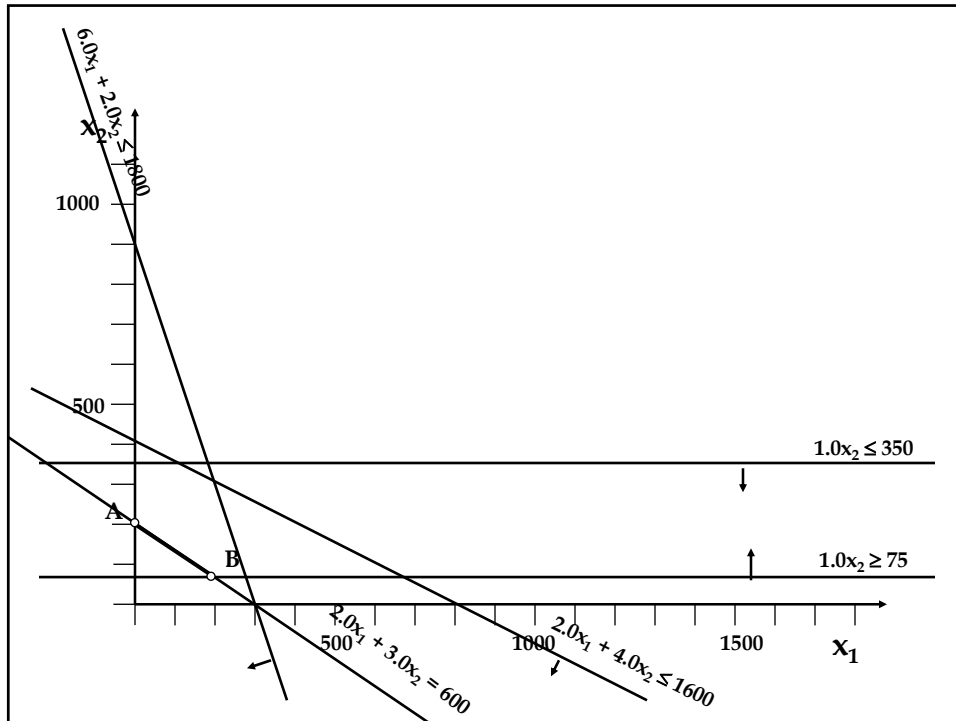
The Extreme Point Theorem (The Fundamental Rule of Linear Programming) - If an optimal solution to a linear programming problem exists, it must occur at one of the extreme points.

Special Cases of Constraints

- a. Equality Constraints - such constraints limit the feasible region to the *line segment represented by the constraint* (why?). For example, if we TV Problem included the constraint**

$$2.0x_1 + 3.0x_2 = 600$$

then two times the value of x_1 plus three times the value of x_2 MUST EQUAL 600 for a solution to be feasible.



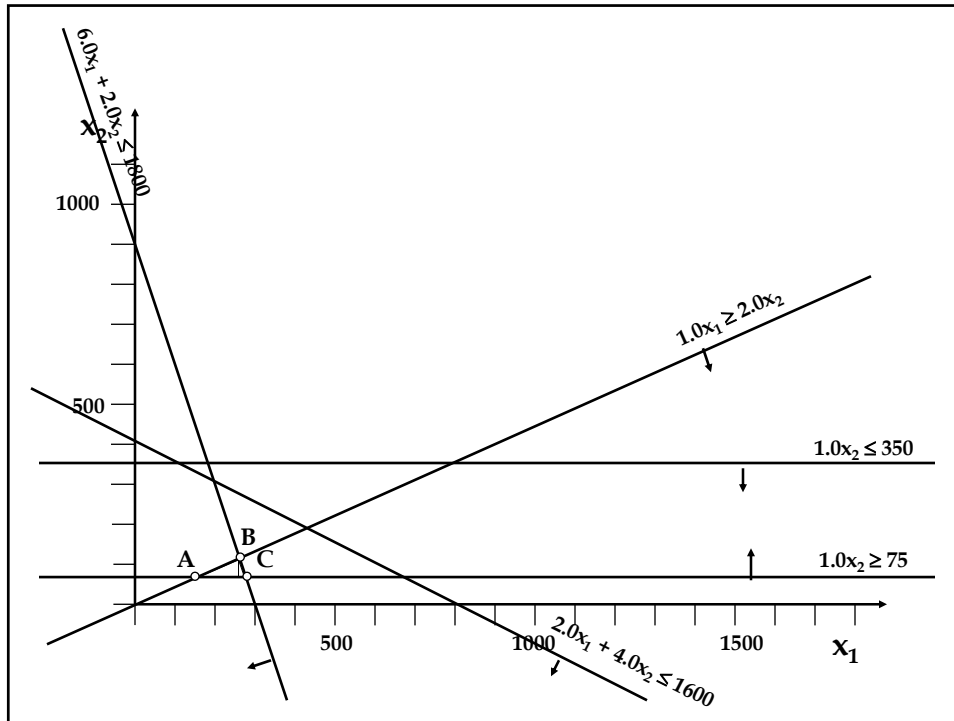
b. Mixture Constraints - if we were given that the number of black & white televisions produced had to be at least twice the number of color sets produced, the resulting constraint could look like:

$$x_1 \geq 2x_2$$

which could be rewritten as

$$x_1 - 2x_2 \geq 0$$

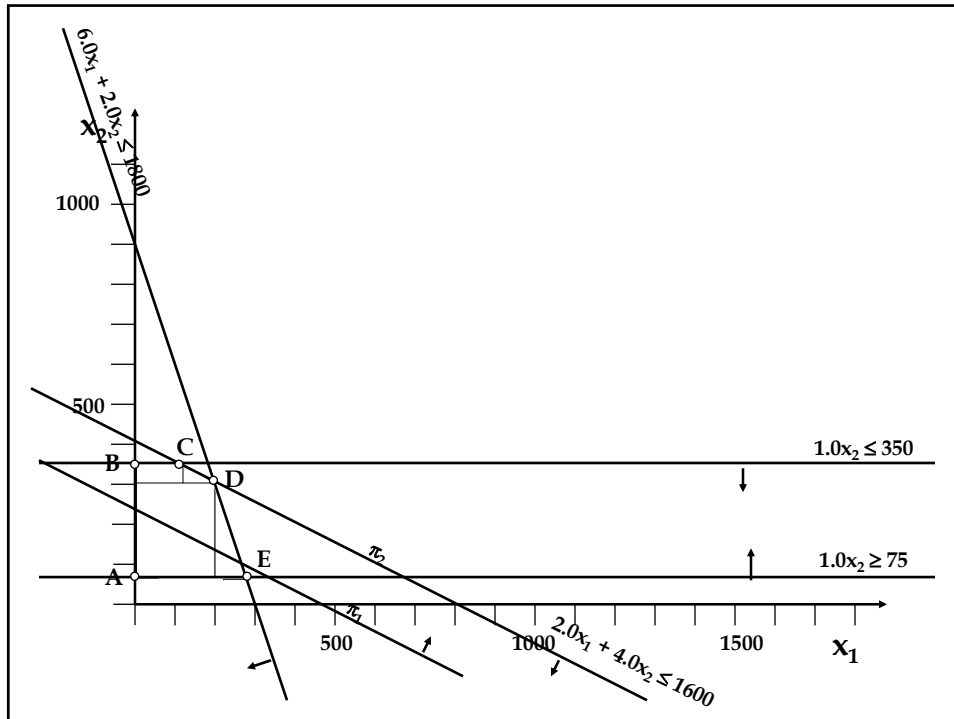
This constraint goes through the origin (why would create some difficulty for us?)



Special Cases of Solutions

- a. **Multiple (Alternate) Optimal Solutions** - more than one extreme point provides an optimal value of the objective function.

This occurs when the objective function is parallel to the last constraint it meets in the feasible region as it moves in the direction of improvement



b. Infeasibility - no set of values for the decision variables is feasible.

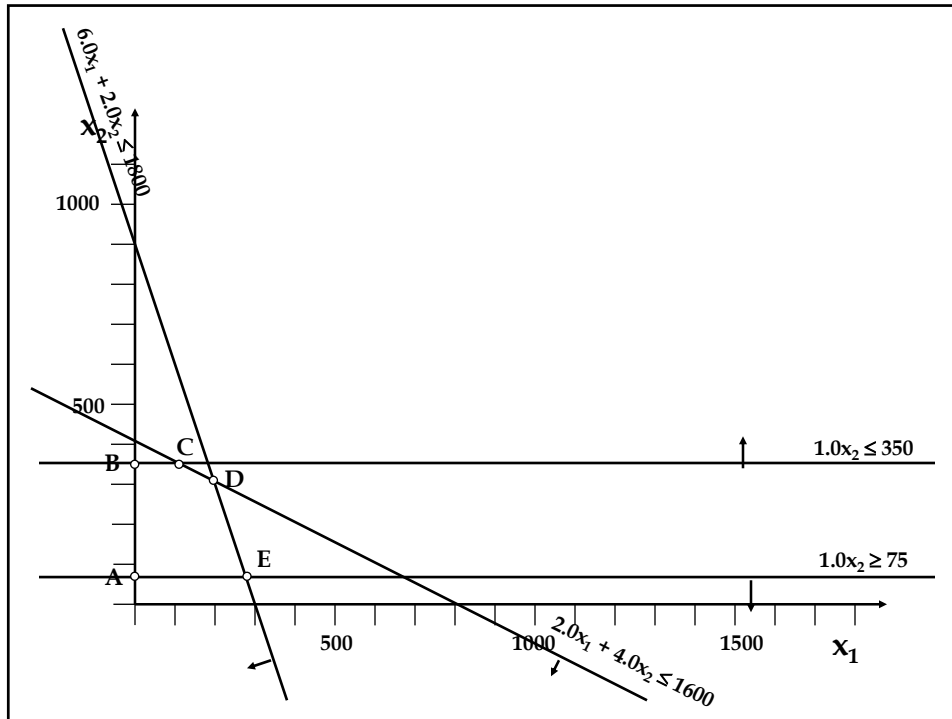
This occurs when the areas represented by the constraints combine to produce NO INTERSECTION.

For example, suppose for the Television Problem that the constraints

$$\begin{aligned} 1.0x_2 &\leq 350 \text{ (\# of available color tubes)} \\ 1.0x_2 &\geq 75 \text{ (minimum \# of color sets)} \end{aligned}$$

were actually

$$\begin{aligned} 1.0x_2 &\leq 75 \text{ (\# of available color tubes)} \\ 1.0x_2 &\geq 350 \text{ (minimum \# of color sets)} \end{aligned}$$



c. Unboundedness - the objective function can move continuously through the feasible region in the direction of improvement without being impeded by constraints.

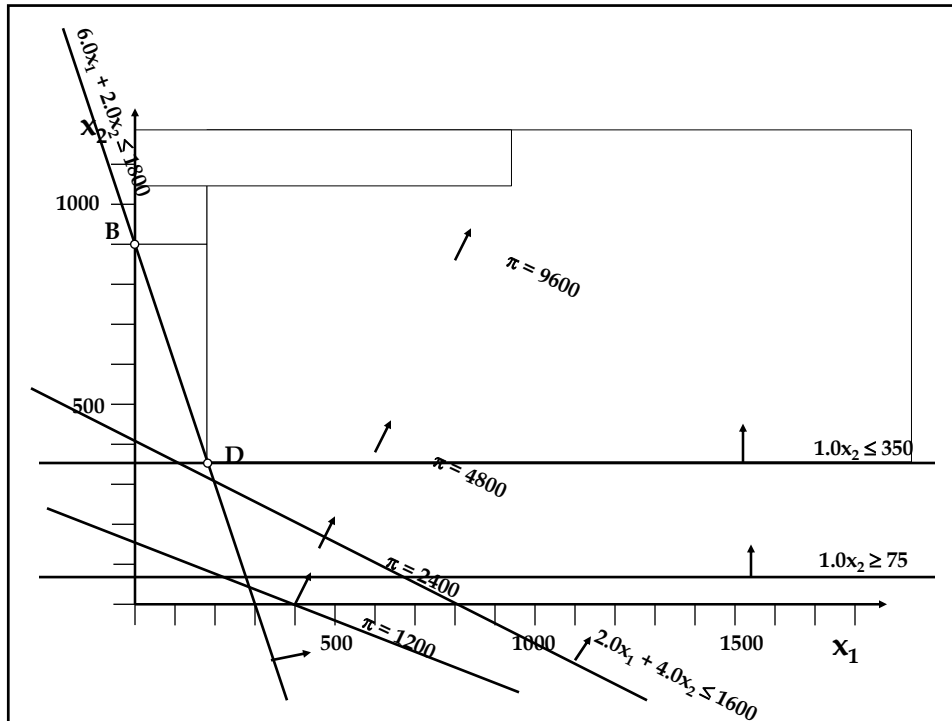
This *usually* occurs when a mistake has been made in formulating the problem.

For example, suppose for the Television Problem that the constraints

- $2.0x_1 + 4.0x_2 \leq 1600$ (# of assembly hours available)
- $6.0x_1 + 2.0x_2 \leq 1800$ (# of fabrication hours available)
- $1.0x_2 \leq 350$ (# of available color tubes)

were actually

- $2.0x_1 + 4.0x_2 \geq 1600$ (# of assembly hours available)
- $6.0x_1 + 2.0x_2 \geq 1800$ (# of fabrication hours available)
- $1.0x_2 \geq 350$ (# of available color tubes)



Slack and Surplus Variables - variables that can be added to or subtracted from the left-hand sides of inequality constraints to convert them to equalities.

- a. **Slack Variable - variable added to the left-hand sides of a less than or equal to constraint to convert it to an equality.**

For example, we can convert the constraint

$$2.0x_1 + 4.0x_2 \leq 1600 \text{ (\# of assembly hours available)}$$

to an equality by adding a slack variable S_L to the left-hand side, i.e.,

$$2.0x_1 + 4.0x_2 + S_L = 1600 \text{ (\# of assembly hours available)}$$

Note that S_L represents unused assembly hours!

b. Surplus Variable - variable subtracted from the left-hand sides of a greater than or equal to constraint to convert it to an equality.

For example, we can convert the constraint

$$1.0x_2 \geq 75 \text{ (minimum \# of color sets)}$$

to an equality by subtracting a surplus variable S_C from the left-hand side, i.e.,

$$1.0x_2 - S_C = 75 \text{ (minimum \# of color sets)}$$

Note that S_C represents number of color sets produced beyond the minimum number required!

Standard Form of an LP Problem - LP problems in which

- i) all decision variables are on the left-hand sides of constraints and**
- ii) all inequality constraints have been converted to equalities through the use of slack and/or surplus variables**

For example, the Television Problem expressed in Standard Form would look like:

maximize $\pi = 3.0x_1 + 8.0x_2$

subject to: $2.0x_1 + 4.0x_2 + S_A = 1600$
 $6.0x_1 + 2.0x_2 + S_F = 1800$
 $1.0x_2 + S_T = 350$
 $1.0x_2 - S_C = 75$
 $x_1, x_2, S_A, S_F, S_T, S_C \geq 0$

Where x_1 is the number of black & white sets produced
 x_2 is the number of color sets produced
 S_A is the slack assembly labor hours
 S_F is the slack fabrication labor hours
 S_T is the slack color tubes
 S_C is the surplus color tubes beyond demand

Now we can see that we really have $m = 4$ equalities and $n = 6$ unknowns. Remember that we can only solve a system of equations simultaneously when the number of unknowns does not exceed the number of equations

Feasible Extreme Point	x_1	x_2	S_A	S_F	S_T	S_C	π
A	0	75	1300	1650	275	0	600
B	0	350	200	1100	0	275	2800
C	100	350	0	500	0	275	3100
D	200	300	0	0	50	225	3000
E	275	75	750	0	275	0	1425

Note that at each extreme point we have $m = 4$ variables with nonzero values (called Basic Variables) and $n - m = 2$ variables with values of zero (called Nonbasic Variables).