

VII. Inventory Models

A. Inventory Models Under Certainty

Inventory Policy - a strategy or set of guidelines for deciding i) at what level of depletion to reorder (the *Order Point*) and ii) how much to reorder to maintain sufficient levels of inventory (the *Order Quantity*) in an attempt to minimize total inventory costs.

Inventory Costs are generally divided into three classes:

- Ordering (fixed) & Procurement (variable)
- Holding & Carrying
- Shortage (or Out-Of-Stock)

1. Economic Order Quantity (EOQ) Model - Inventory model for constant demand and instantaneous supply replenishment with no backorders.

The parameters of the EOQ Model are:

k = fixed cost per order

A = annual (or periodic) number of items demanded

c = (variable) cost to procure one unit

h = annual (or periodic) cost *per dollar* value of holding inventory

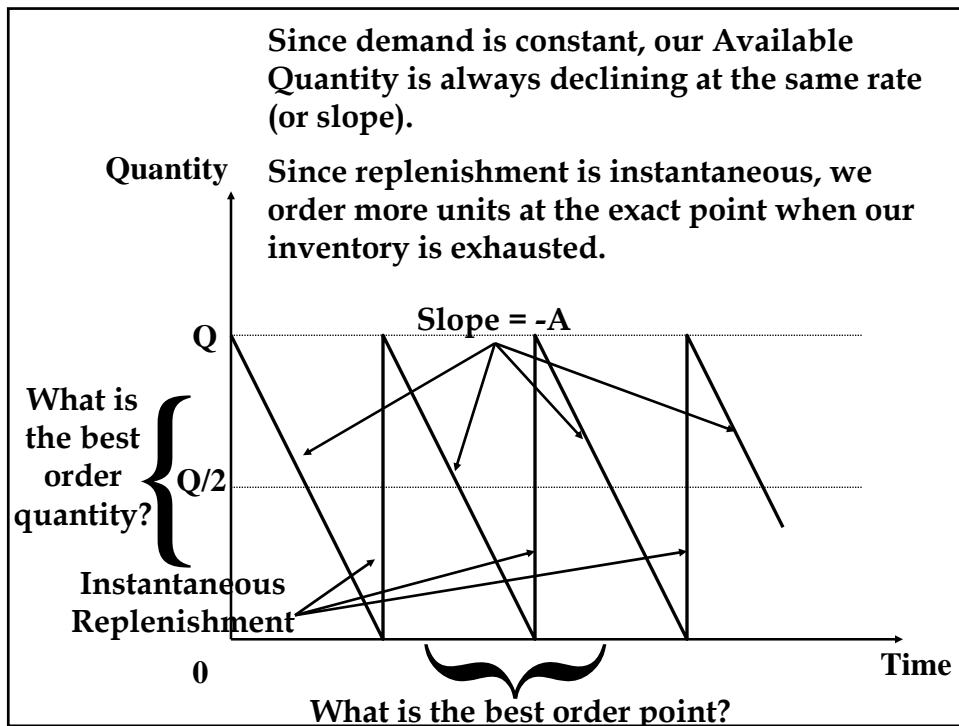
T = time between orders

Again, the objective is to choose the number of items to order (order quantity)

Q = Order Quantity

that minimizes

Total Annual Cost = Ordering + Holding + Procurement Costs



To find the optimal order point and quantity minimize:

Total Annual Cost = Ordering + Holding + Procurement Costs

$$TC(Q) = \frac{A}{Q}k + hc\frac{Q}{2} + Ac$$

Let's dissect these components individually:

- Annual (Periodic) Ordering Costs: we have

$\frac{A}{Q}$ is the number of orders placed annually (periodically)

so $\frac{A}{Q}k$ is the Annual (Periodic) Ordering Cost

- Annual (Periodic) Holding Costs: we have

$\frac{Q}{2}$ is the mean inventory (because demand is constant)

hc is the annual (periodic) cost of holding one unit in inventory

so $hc\frac{Q}{2}$ is the Annual (Periodic) Holding Cost

- Annual (Periodic) Procurement Costs: we have

A_c is the annual (periodic) procurement cost

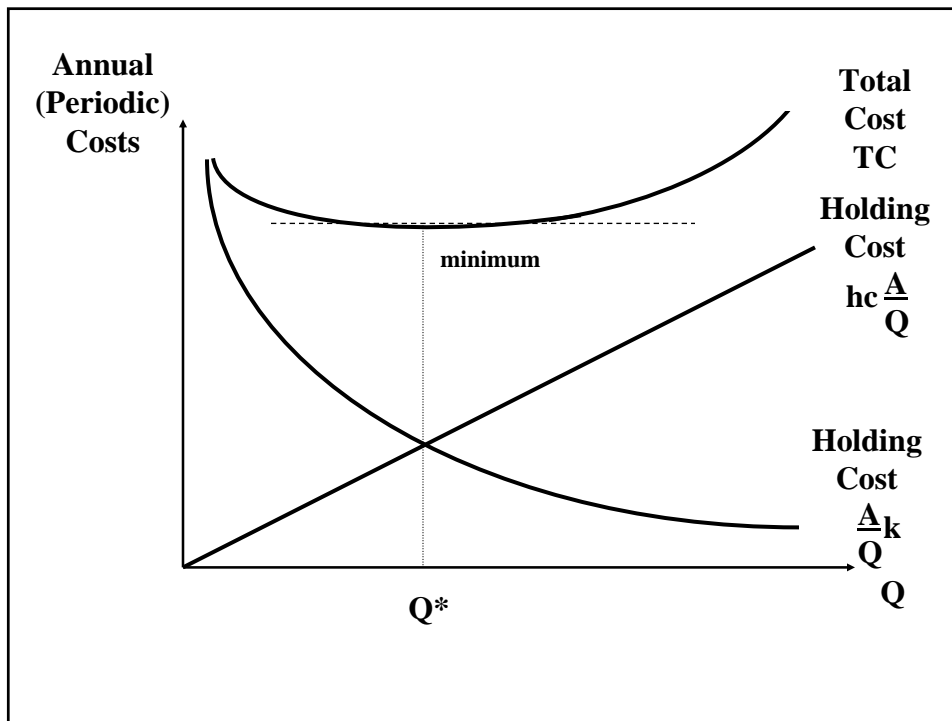
Thus we have that

Total Annual Cost = Ordering + Holding + Procurement Costs

$$TC(Q) = \frac{A_k}{Q} + hc\frac{Q}{2} + A_c$$

Since Procurement Costs are not related to inventory levels, it is customary to ignore A_c in the EOQ model.

Also Note that $TC()$ is a function and Q is the argument - the goal is to find the value of Q (our decision variable) that minimizes $TC()$ (our objective function).



Differentiating our function

$$TC(Q) = \frac{A}{Q}k + hc\frac{Q}{2}$$

with respect to Q yields:

$$Q^* = \sqrt{\frac{2Ak}{hc}}$$

which is often referred to as the Optimal Order Quantity or Wilson's formula (for R. H. Wilson).

Now Q^* can be used to determine the Optimal Order Point

$$T^* = \frac{Q^*}{A}$$

We now have a complete Inventory Policy.

Example: *We're Cookin'* is a local restaurant supply store. The store stocks a particular type of commercial food processor that is very popular among its customers, with expected sales of 1000 units during the next year. *We're Cookin'* pays \$100.00 in shipping and handling costs every time they order additional units of this food processor. In addition, they pay a wholesale cost of \$250.00 per unit for this food processor. Finally, *We're Cookin's* warehouse manager has determined that the costs of handling, warehouse space, and interest result in an annual cost of \$2.00 per dollar of value to hold one unit in inventory. We wish to find the optimal order point and quantity.

- First we identify the parameters:

k (fixed cost per order) = 100

A (annual number of items demanded) = 1000

c (unit cost to procure one unit) = 250

h (annual cost per dollar value of holding inventory) = 2

- Now we use this information ($k = 100$, $A = 1000$, $c = 250$, and $h = 2$) to find:

$$Q^* = \sqrt{\frac{2Ak}{hc}}$$

by simple substitution we have:

$$Q^* = \sqrt{\frac{2(1000)(100)}{(2)(250)}} = 20.0$$

which is the optimal EOQ Order Quantity.

- Now we use this information ($Q^* = 20.0$) to find:

$$T^* = \frac{20.0}{1000} = 0.020 \text{ years}$$

which is the optimal EOQ Order Point (i.e., every 7.30 days).

Limitations of the EOQ Model

- Assumes constant demand;
- Assumes instantaneous supply;
- Allows for no backorders;
- Can only be applied to multicommodity situations when demand for each commodity is independent inventory space is unlimited.

2. We can make a slight modification to the EOQ model to allow for backorders - we simply introduce a new parameter p that represents a shortage (or out-of-stock) penalty:

$$TC(Q,S) = \frac{A_k}{Q} + \frac{hcS^2}{2Q} + \frac{p(Q-S)^2}{2Q}$$

Note that S is the number of reorder units that will be devoted to inventory ($Q - S$ is used to fill backorders).

Again we ignore Procurement Costs Ac (which are not relevant to inventory decisions). A little differential calculus and some algebra applied to the result:

$$TC(Q, S) = \frac{A}{Q}k + hc \frac{A}{Q}$$

yields the following equations:

$$Q^* = \sqrt{\frac{2Ak}{hc}} \sqrt{\frac{p+hc}{p}} \quad \text{and} \quad S^* = \sqrt{\frac{2Ak}{hc}} \sqrt{\frac{p}{p+hc}}$$

Again Q^* can be used to determine the Optimal Order Point

$$T^* = \frac{Q^*}{A}$$

We now have a complete Inventory Policy that considers backorders.

Example: Suppose the owner of *We're Cookin'* has determined that he loses \$5.00 every time a customer comes in for the commercial food processor and he is out of stock. What would be the revised optimal order point and quantity.

- First we identify the parameters:

k (fixed cost per order) = 100

A (annual number of items demanded) = 1000

c (cost to procure one unit) = 250

h (annual cost per dollar value of holding inventory) = 2

and now

p (per unit shortage penalty) = 5

- Now we use this information ($k = 100$, $A = 1000$, $c = 250$, $h = 2$, and $p = 5$) to find:

$$Q^* = \sqrt{\frac{2Ak}{hc}} \sqrt{\frac{p+hc}{p}}$$

by simple substitution we have:

$$Q^* = \sqrt{\frac{2(1000)(100)}{(2)(250)}} \sqrt{\frac{5+(2)(250)}{5}} = 200.998$$

which is the optimal EOQ Order Quantity.

- Now we use this information ($Q^* = 200.998$) to find:

$$T^* = \frac{200.998}{1000} = 0.201 \text{ years}$$

which is the optimal EOQ Order Point (i.e., every 73.36 days).

The optimal inventory policy is now to order 204.94 units every 74.80 days. The number of units to be stored in inventory from each order is

$$S^* = \sqrt{\frac{2(1000)(100)}{(2)(250)}} \sqrt{\frac{5}{5+(2)(250)}} = 1.990$$

The remaining $Q^* - S^* = 200.99 - 1.99 = 199.01$ units will be devoted to the satisfaction of existing backorders.

Limitations of the EOQ Model with Backorders in addition to other limitations of the EOQ Model (assumes constant demand and instantaneous supply; can only be applied to multicommodity situations when demand for each commodity is independent inventory space is unlimited):

- Assumes constant shortage penalty;
- Assumes no fixed shortage-cost component.

3. Service Level Based Inventory Policies

For a *Service Level Proportion L* (proportion of customers we wish to satisfy *immediately*), we wish to choose the Order Quantity Q and Service Level S so that

$$\frac{Q^* - S^*}{Q^*} = 1 - L \quad \text{or} \quad Q^* - S^* = (1 - L)Q^*$$

Again a little algebra can be used to show that

$$LQ^* = S^*$$

So by substitution of the EOQ Model formulas for Q^* and S^* we have

$$L\sqrt{\frac{2Ak}{hc}}\sqrt{\frac{p+hc}{p}} = \sqrt{\frac{2Ak}{hc}}\sqrt{\frac{p}{p+hc}}$$

A little more algebra can be used to derive the Imputed Service Level:

$$p = \frac{hcL}{1 - L}$$

Example: Suppose the owner of *We're Cookin'* has wants to fill 80% of all customer demand immediately. What would be the revised optimal order point and quantity.

- First we recognize that we continue to use the same parameters:

k (fixed cost per order) = 100

A (annual number of items demanded) = 1000

c (cost to procure one unit) = 250

h (annual cost per dollar value of holding inventory) = 2

but now must recalculate p (the per unit shortage penalty)

$$p = \frac{hcL}{1 - L} = \frac{(2)(250)(0.80)}{1 - 0.80} = 2000$$

This value is now substituted into the EOQ formulas for Q^* and S^* :

$$Q^* = \sqrt{\frac{2Ak}{hc}} \sqrt{\frac{p+hc}{p}}$$

by simple substitution we have:

$$Q^* = \sqrt{\frac{2(1000)(100)}{(2)(250)}} \sqrt{\frac{2000+(2)(250)}{2000}} = 22.361$$

which is the optimal EOQ Order Quantity.

- Now we use this information ($Q^* = 22.361$ to find:

$$T^* = \frac{22.361}{1000} = 0.0224 \text{ years}$$

which is the optimal EOQ Order Point (i.e., every 8.16 days).

The optimal inventory policy is now to order 50.00 units every 18.25 days. The number of units to be stored in inventory from each order is

$$S^* = \sqrt{\frac{2(1000)(100)}{(2)(250)}} \sqrt{\frac{2000}{2000+(2)(250)}} = 17.89$$

The remaining $Q^* - S^* = 22.36 - 17.89 = 4.47$ units will be devoted to the satisfaction of existing backorders.

This model is essentially the EOQ Model with Backorders and so has the same limitations (assumes constant demand and instantaneous supply; can only be applied to multicommodity situations when demand for each commodity is independent inventory space is unlimited; assumes constant shortage penalty and no fixed shortage-cost component).

4. Economic Production Quantity (EPQ) Model - Inventory model for constant demand and constant rate of production (supply replenishment) with no backorders.

The parameters of the EOQ Model are:

k = fixed cost per order

A = annual (or periodic) number of items demanded

B = annual (or periodic) number of items produced

c = (variable) cost to procure one unit

h = annual (or periodic) cost per dollar value of holding inventory

T = time between orders

Again, the objective is to choose the number of items to order (order quantity)

Q = Order Quantity

that minimizes

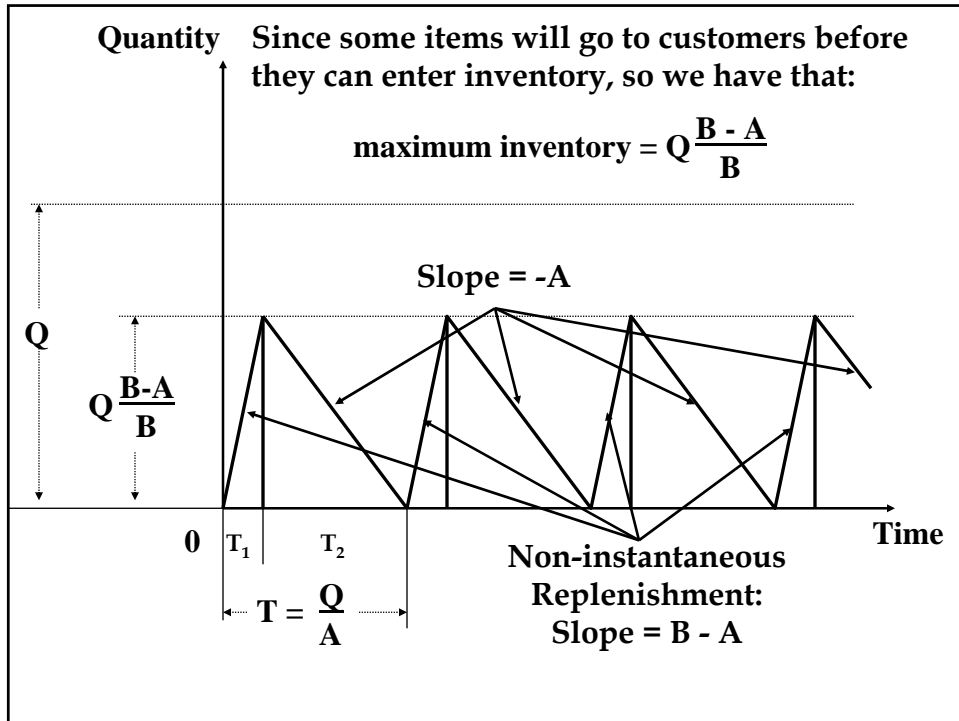
Total Annual Cost = Ordering + Holding + Procurement Costs

In such an inventory system, each inventory system is divided into two phases:

- Production Phase
- Demand-Only Phase

Since some items will go to customers before they can enter inventory, so we have that:

$$\text{maximum inventory} = Q \frac{B - A}{B}$$



The durations (T_1, T_2) of the two phases are:

$$T_1 = \frac{Q}{B}$$

and since $T = \frac{Q}{A}$

we have that

$$T_2 = T - T_1 = \frac{Q}{A} - \frac{Q}{B} = Q \frac{B-A}{AB}$$

The total cost function becomes:

$$TC(Q) = \frac{A}{Q}k + hc \frac{Q}{2} \frac{B-A}{B}$$

The optimal order quantity becomes:

$$Q^* = \sqrt{\frac{2Ak}{hc}} \sqrt{\frac{B}{B-A}}$$

Example: Suppose the owner of *We're Cookin'* produces the commercial food processors at a rate of 1500 annually. What would be the revised optimal order point and quantity.

- First we recognize that we continue to use the same parameters:

k (fixed cost per order) = 100

A (annual number of items demanded) = 1000

c (cost to procure one unit) = 250

h (annual cost per dollar value of holding inventory) = 2

but now we also have

B = (annual number of units that can be produced) = 1500

The optimal order quantity is:

$$Q^* = \sqrt{\frac{2Ak}{hc}} \sqrt{\frac{B}{B-A}} = \sqrt{\frac{2(1000)(100)}{(2)(250)}} \sqrt{\frac{1500}{1500-1000}} = 34.641$$

and the durations (T_1 , T_2) of the two phases are:

$$T_1^* = \frac{Q^*}{B} = \frac{34.641}{1500} = 0.0231$$

and since $T^* = \frac{Q^*}{A}$

we have that

$$T_2^* = T^* - T_1^* = \frac{34.64}{1000} - 0.0231 = 0.0115$$

or $T_1 = 8.43$ days and $T_2 = 4.21$ days.

B. Inventory Models Under Uncertainty (i.e., Stochastic Inventory Models)

We now explore ways to deal with development of Inventory Policy when some parameter(s) of the inventory model (such as demand) are represented by a probability distribution (are uncertain).

1. The Newsvendor (Single Period Inventory) Problem - useful when demand is uncertain and the product is perishable.

Since we are stocking inventory only once during a period (at the beginning of the period), sales will be one of two values (given D = periodic demand and Q = stocked inventory at the beginning of the period):

$$\text{Sales} = \begin{cases} D & \text{if } D \leq Q \\ Q & \text{if } D > Q \end{cases}$$

The parameters (cost elements) of the Newsvendor Model are:

c = (variable) cost to procure one unit

h_E = additional cost of each item held at the end of the inventory cycle

p_S = penalty for each item short (i.e., value of lost goodwill)

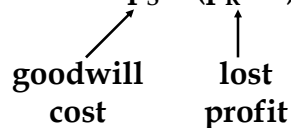
p_R = selling price

From these parameters we can easily derive:

Total Cost per Leftover Item = $h_E + c$

Total Cost per Item Short = $p_S + (p_R - c)$

goodwill cost lost profit



If demand for some inventory cycle is at some level d , then we have that:

$$TC(Q) = cd + \begin{cases} (h_E + c)(Q - d) & \text{if } d \leq Q \\ (p_S + p_R - c)(d - Q) & \text{if } d > Q \end{cases}$$

The objective now should be the choice of Q that minimizes the function $TC(Q)$, i.e., minimizes total cost.

In order to achieve this we first need to calculate the *Expected Demand*:

$$\mu = E(D) = \sum_d d \Pr [D=d]$$

This is then used to calculate the *Total Expected Cost*

$$TC(Q) = c\mu + \sum_{d=0}^Q (h_E + c)(Q - d) \Pr [D=d] + \sum_{d>Q} (p_S + p_R - c)(d - Q) \Pr [D=d]$$

↑ goodwill cost ↑ goodwill cost ↑ lost profit

Example: Suppose the owner of *We're Cookin'* stocks and sells fresh *celery root*, a relatively rare vegetable with a one-week shelf life. They pay \$0.30 for celery root and can sell the celery root for \$0.50 each. Celery root is stocker once weekly - any not sold in one week's time is discarded at a cost of \$0.01 each, but the Manager believes that a shortage results in a loss of \$0.05 per unit of unfulfilled demand. From past data the Manager has estimated that the probability distribution for weekly celery root demand is:

<u>Possible Demand (d)</u>	<u>Probability of d, i.e., Pr[D = d]</u>
12	0.10
13	0.15
14	0.30
15	0.25
16	0.15
17	<u>0.05</u>
	1.00

We must first find the Expected Value of weekly fresh celery root sales:

Possible Demand (d)	Probability of d, i.e., Pr[D = d]	d*Pr[D = d]
12	0.10	0.12
13	0.15	1.95
14	0.30	4.20
15	0.25	3.75
16	0.15	2.40
17	<u>0.05</u>	<u>0.85</u>
	1.00	$\mu = E(D) = 14.35$

Notice that this is not the same as

$$\frac{\sum d}{n} = \frac{87}{6} = 14.5$$

Why?

A table of our final calculations (performed in the Excel spreadsheet *NEWSVENDOR.xls*) could look like this:

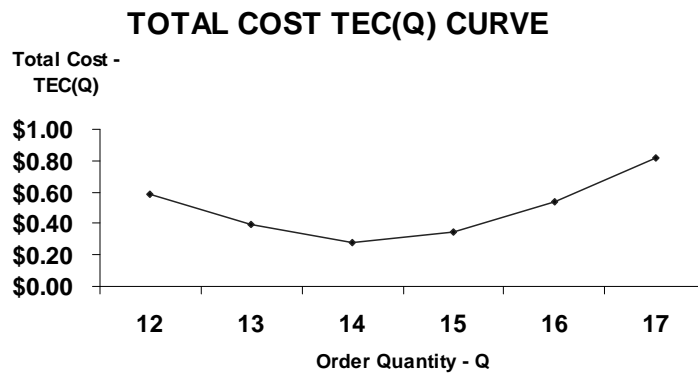
Possible Demand (d)	Probability Pr[D=d]	Demand x Probability d Pr[D=d]	Holding Cost		Shortage Cost	
			cost (0.3 + 0.011)(15 - d)	cost x probability	cost (0.05 + 0.5 - 0.3)(d - 15)	cost x probability
12	0.10	1.20	\$0.93	\$0.09	-	-
13	0.15	1.95	\$0.62	\$0.09	-	-
14	0.30	4.20	\$0.31	\$0.09	-	-
15	0.25	3.75	\$0.00	\$0.00	-	-
16	0.15	2.40	-	-	\$0.25	\$0.04
17	0.05	0.85	-	-	\$0.50	\$0.03
	1.00	14.35		\$0.28		\$0.06

Note that we have used this Excel worksheet to calculate TEC(15) or the Total Expected Cost of this inventory system at an order quantity of Q = 15. The spreadsheet is capable of calculating TEC(Q) for any value of Q...

If we find $TEC(Q)$ for each possible value of Q (12, 13,...,17), we can plot a Total Cost Curve:

Order Quantity (Q)	Total Cost TEC(Q)
12	\$0.5875
13	\$0.3935
14	\$0.2835
15	\$0.3415
16	\$0.5395
17	\$0.8215

Notice that Purchase Costs (Qc) have been omitted here as irrelevant.



Notice the (even though slight) convexity - there is a minimum cost order quantity at $Q = 14$.

This property (convexity) lends the results to a marginal analysis - calculate

$$\text{Difference} = \text{TEC}(Q + 1) - \text{TEC}(Q)$$

Order Quantity (Q)	Total Cost TEC(Q)	Marginal Cost TEC(Q +1) - TEC (Q)
12	\$0.5875	-
13	\$0.3935	(\$0.1940)
14	\$0.2835	(\$0.1100)
15	\$0.3415	\$0.0580
16	\$0.5395	\$0.1980
17	\$0.8215	\$0.2820

The optimal point is the largest Q with a negative marginal cost (Q = 14 in this case).

However, we can use our previous results to derive an Optimal Order Quantity for this scenario:

Q^* is the smallest possible demand such that:

$$\Pr[D \leq Q^*] \geq \frac{p_S + p_R - c}{(p_S + p_R - c) + (h_E + c)}$$

for our problem we have:

$$\frac{p_S + p_R - c}{(p_S + p_R - c) + (h_E + c)} = \frac{.05 + .50 - .30}{(.05 + .50 - .30) + (.01 + .30)} = 0.446$$

and the cumulative distribution of D is

Possible Demand (d)	Probability Pr[D = d]	Cumulative Probability Pr[d ≤ D]
12	0.10	0.10
13	0.15	0.25
14	0.30	0.55
15	0.25	0.80
16	0.15	0.95
17	0.05	1.00
	1.00	

Again, the optimal strategy is $Q^* = 14$.

We can use this information to derive additional insights into the nature of an inventory system:

The *Expected Shortage* we will experience over the inventory cycle is:

$$B(Q) = \sum_{d>Q} (d - Q) \Pr [D=d]$$

while the *Expected Surplus* we will experience over the inventory cycle is:

$$Q - \mu + B(Q) = \sum_{d=0}^Q (Q - d) \Pr [D=d]$$

which leads to an alternative (but equivalent) formula for *Total Expected Cost*:

$$TEC(Q) = c\mu + (h_E + c)[Q - \mu + B(Q)] + (p_S + p_R - c)B(Q)$$

The *Expected Shortage* and *Expected Surplus* we will experience over the inventory cycle are:

Possible Demand (d)	Probability Pr[D=d]	Expected Shortage (d - Q)Pr[D = d]	Expected Surplus (d - Q)Pr[D = d]
12	0.10	-	0.30
13	0.15	-	\$0.30
14	0.30	-	\$0.30
15	0.25	-	\$0.00
16	0.15	0.15	-
17	0.05	0.10	-
	1.00	0.25	0.90

which again leads to the *Total Expected Cost*:

$$TEC(Q) = c\mu + (h_E + c)[Q - \mu + B(Q)] + (p_S + p_R - c)B(Q)$$

2. The Newsvendor (Single Period Inventory) Problem with Continuous Demand - useful when demand is uncertain and is a continuous variable while the product is perishable.

The only real change to the way we approached the original Newsvendor problem is that we choose Q such that:

$$\Pr[D \leq Q^*] = \frac{p_S + p_R - c}{(p_S + p_R - c) + (h_E + c)}$$

Example: Suppose weekly sales of celery root at *We're Cookin'* are normally distributed with a mean of 15 and standard deviation of 1.5. Given the previous problem parameters, how many celery roots should *We're Cookin'* order per week?

We previously calculated the ratio:

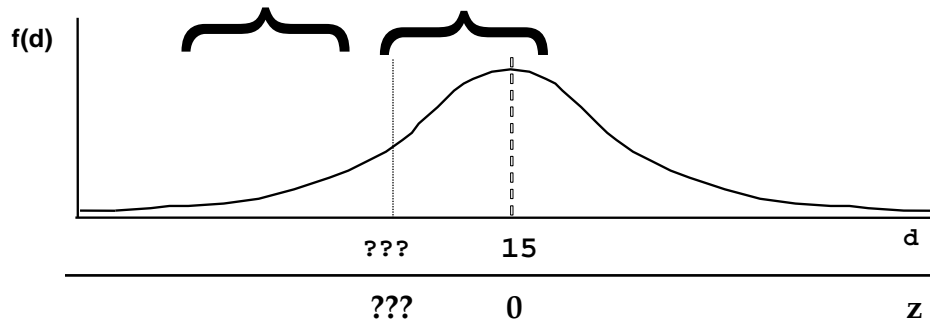
$$\frac{p_S + p_R - c}{(p_S + p_R - c) + (h_E + c)} = \frac{.05 + .50 - .30}{(.05 + .50 - .30) + (.01 + .30)} = 0.446$$

Thus we want to order the quantity Q such that 44.6% of the normal distribution lies beneath, i.e.,

What is the value of d such that $P(D \leq d) = 0.446$?

Area of Probability = 0.446

Find a value of z for this Area (.054) using Standard Normal Table



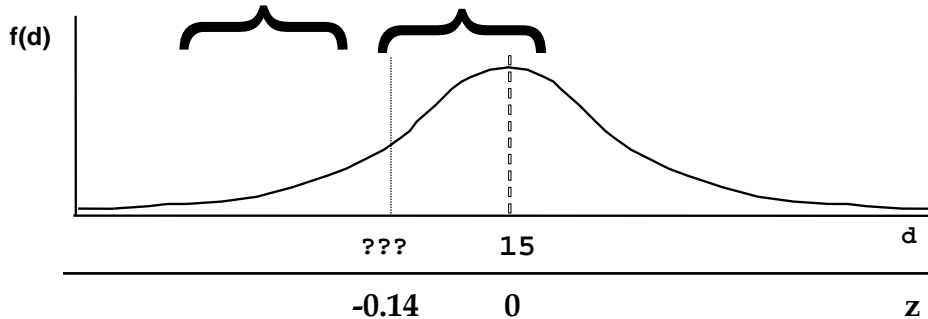
The Standard Normal Distribution (Appendix A, Table 1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0998	0.0958	0.0918	0.0877	0.0837	0.0796	0.0756	0.0715	0.0675	0.0635
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

What is the value of d such that $P(D \leq d) = 0.446$?

Area of Probability = 0.446

Find a value of z for this Area using Standard Normal Table



Then use $Q^* = \mu + z\sigma = 15 + (-0.14)1.5 = 14.79$.

For This model (with normally distributed demand) the *Expected Shortage* is:

$$B(Q) = \begin{cases} \sigma L\left(\frac{Q - \mu}{\sigma}\right) & Q \geq \mu \\ \mu - Q + \sigma L\left(\frac{\mu - Q}{\sigma}\right) & Q < \mu \end{cases}$$

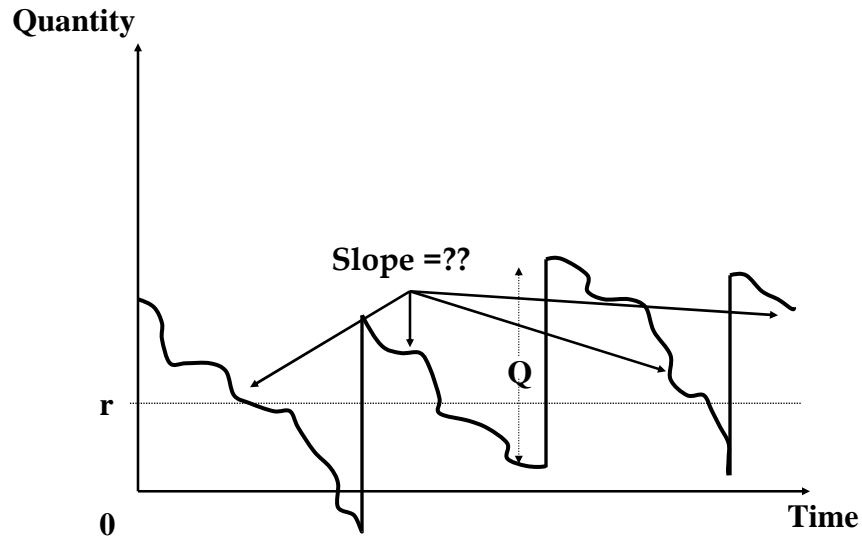
where $L(\)$ represents the Normal Loss Function (values in Appendix C). For example, in our problem for $Q = 13$ we have:

$$\begin{aligned} B(Q) &= \mu - Q + \sigma L\left(\frac{\mu - Q}{\sigma}\right) \quad (\text{because } Q < \mu) \\ &= 14.35 - 13 + 1.5L\left(\frac{14.35 - 13}{1.5}\right) = 14.35 - 13 + 1.5L(0.9) = 1.5006 \end{aligned}$$

In our problem for $Q = 16$ we have:

$$\begin{aligned} B(Q) &= \sigma L\left(\frac{Q - \mu}{\sigma}\right) \quad (\text{because } Q \geq \mu) \\ &= 1.5L\left(\frac{16 - 13}{1.5}\right) = 1.5L(2.0) = 0.1029 \end{aligned}$$

3. **Multiperiod Inventory Problems** - we can/must now consider demand and lead time required to fill orders. - The basis of such decisions are referred to as (r, Q) policies (where r now represents the *Reorder Point*).



Because we are now dealing with multiple periods, we return to the Economic Order Quantity (EOQ) framework.

Recall that the parameters of the EOQ Model are:

k = fixed cost per order

A = annual (or periodic) number of items demanded

c = (variable) cost to procure one unit

h = annual (or periodic) cost *per dollar* value of holding inventory

T = time between orders

Note that annual Demand (A) is now uncertain - we adjust by redefining A as:

A = mean annual (or periodic) demand

And again, the objective is to choose the number of items to order (order quantity)

Q = Order Quantity

that minimizes

Total Annual Cost = Ordering + Holding + Procurement Costs

Since the Mean Annual Demand is constant (individual period demands fluctuate) we can define:

$\frac{A}{Q}k$ is the Expected Annual (Periodic) Ordering Cost

We have to be concerned with the uncertainty of demand (and particularly the demand during lead time or *Lead Time Demand*). Denote the probability that d units will be demanded during lead time L by

$$\Pr_L[D = d]$$

which is uncertain in both i) the quantity of demand and ii) the length of lead time. We address these two uncertainties simultaneously by focusing on the expected value of D, i.e.,

μ = Mean Lead Time Demand

We now have two major considerations in inventory evaluation (based on how shortages are handled):

- Backordering: all demand experienced during out-of stock is filled when stock arrives (with a shortage penalty)
- Lost Sales: all demand experienced during out-of stock is lost (with a different shortage penalty)

Multiperiod Inventory Problems *with Backordering*:
under these circumstances we have:

$$B(r) = \sum_{d>r} (d - r)Pr_L [D=d]$$

This leads to the Total Annual Expected Cost:

$$TEC(r, Q) = \frac{A}{Q}k + hc\left(\frac{Q}{2} + r - \mu\right) + p_s\frac{A}{Q}B(r)$$

Now the objective is to find the values of Q and r that simultaneously minimize the Total Estimated Cost function TEC(). We can find the optimal reorder point (r*) if we know Q - it is the smallest quantity having a cumulative lead-time demand probability meeting the *Condition for Reorder Point*:

$$Pr_L [D \leq r^*] \geq 1 - \frac{hcQ}{p_sA}$$

We can also find the optimal reorder quantity (Q*) if we know r:

$$Q^* = \sqrt{\frac{2A[k + p_sB(r)]}{hc}}$$

Since we need to know Q to find r* (and we need to know r to find Q*), we often iterate between the two formulas (using the results of the most recent calculation for one of these decision variables to find the value of the other).

Wilson's formula provides a good starting point:

$$Q_1 = \sqrt{\frac{2Ak}{hc}}$$

The subscript 1 indicates that this is a starting value (I.e., first iteration) for Q.

Example: Again consider the plight of *We're Cookin'* - they stock and sell fresh *celery root*, a vegetable with a one-week shelf life. They pay \$0.30 for celery root and can sell the celery root for \$0.50 each. Celery root is stocked once weekly - any not sold in one week's time is discarded at a cost of \$0.01 each, but the Manager believes that a shortage results in a loss of \$0.05 per unit of unfulfilled demand. From past data the Manager has estimated that the probability distribution for weekly celery root demand is:

<u>Possible Demand (d)</u>	<u>Probability of d, i.e., Pr[D = d]</u>
12	0.10
13	0.15
14	0.30
15	0.25
16	0.15
17	<u>0.05</u>
	1.00

We use Wilson's formula to start our first iteration (to find a starting Q):

$$Q_1 = \sqrt{\frac{2Ak}{hc}} = \sqrt{\frac{2(1000)(100)}{2(250)}} = 200.998$$

So we use $Q = 201$ to compute a new reorder point r:

$$\Pr_L [D \leq r^*] \geq 1 - \frac{hcQ}{p_s A} (=0.9879)$$

Thus our reorder point at this iteration is $r = 17$:

Possible Demand (d)	Probability of d, i.e., $\Pr[D = d]$	Cumulative Probability, i.e., $\Pr[D \leq d]$
12	0.10	0.10
13	0.15	0.25
14	0.30	0.55
15	0.25	0.80
16	0.15	0.95
17	<u>0.05</u>	1.00
	1.00	

We now use this information to find a new value for Q :

$$Q^* = \sqrt{\frac{2A[k + p_s B(r)]}{hc}}$$

And continue until the reorder point r converges (to r^*).

Multiperiod Inventory Problems *with Lost Sales*: under these circumstances we have Total Annual Expected Cost:

$$TEC(r, Q) = \frac{A'k}{Q} + hc \left(\frac{Q}{2} + r - \mu + B(r) \right) + (p_s + p_R - c) \frac{A'}{Q} B(r)$$

where $A' = A(1 - B(r)/Q)$.

Again the objective is to find the values of Q and r that simultaneously minimize the Total Estimated Cost function $TEC()$. And again we can find the optimal reorder point (r^*) if we know Q - it is the smallest quantity having a cumulative lead-time demand probability meeting the *Condition for Reorder Point*:

$$\Pr_L [D \leq r^*] \geq \frac{(p_s + p_R - c)A}{hcQ + (p_s + p_R - c)A}$$

We can also find the optimal reorder quantity (Q^*) if we know r :

$$Q^* = \sqrt{\frac{2A[k + (p_S + p_R - c)B(r)]}{hc}}$$

We can again use an iterative procedure and use Wilson's formula for the starting point:

$$Q_1 = \sqrt{\frac{2Ak}{hc}}$$